UNIVERSIDAD DE MURCIA
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A separable Banach lattice that contains isometric copies of all others

Un retículo de Banach separable que contiene copias isométricas de todos los demás

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Declaración de Originalidad

Begoña García Malaxechebarría, autora del Trabajo de Fin de Grado "A separable Banach lattice that contains isometric copies of all others", bajo la tutela del profesor Antonio Avilés López y de José David Rodríguez Abellán, declara que el trabajo que presenta es original, en el sentido de que ha puesto el mayor empeño en citar debidamente todas las fuentes utilizadas, y que la obra no infringe el copyright de ninguna persona.

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En la Secretaría de la Facultad de Matemáticas se ha presentado una copia firmada de esta declaración.

Me gustaría empezar esta memoria diciendo que ha sido un gran honor tener la oportunidad de trabajar con Antonio Avilés López y José David Rodríguez Abellán. Les agradezco infinitamente a ambos sus consejos, toda la ayuda desinteresada que me han prestado y el conocimiento que he adquirido con esta experiencia.

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Contents

Resumen

En este resumen se pretende hacer una introducción al contenido de esta memoria con la intención de plantear al lector una idea general de la misma. Aunque resulta complicado resumir algo tan extenso, se pretenden mencionar los conceptos más importantes con el fin de ilustrar lo máximo posible cada uno de los capítulos que la constituyen haciendo un breve recorrido por cada uno de ellos. Puesto que en el programa del Grado en Matemáticas de la Universidad de Murcia no consta el estudio de los retículos de Banach, en este trabajo se parte de una base que se asume ha sido adquirida en asignaturas como *Topología de Espacios Métricos* o *Análisis Funcional* y se definen, a partir de ahí, todas las demás nociones necesarias. A pesar de ello, parece coherente y lógico repasar todos aquellos criterios básicos que se suponen ya conocidos en un primer capítulo introductorio para facilitar su comprensión.

El principal objetivo de esta memoria consiste en exponer un retículo de Banach separable para el cual cualquier otro se puede identificar como una copia isométrica suya, es decir, un retículo de Banach universal inyectivo para la clase de los retículos de Banach separables. Este resultado, detallado en el Capítulo 4 mediante un profundo análisis de la demostración que se puede encontrar en el artículo *Separable Universal Banach Lattices* [Leung et al., 2019], es fundamental pues nos permite ser capaces de comparar un retículo de Banach cualquiera con uno particular mediante el orden dado por la inclusión. Antes de enfrentar la prueba del ya citado artículo, se presenta una contextualización en conceptos como retículo vectorial y retículo de Banach, y una serie de propiedades imprescindibles para su demostración. Además, he considerado de gran interés mostrar primero una versión equivalente para espacios de Banach, el Teorema de Banach-Mazur, que plantea la necesidad de desarrollarlo más concretamente para retículos de Banach.

Mediante el **Capítulo 1** se hace una introducción a conceptos razonablemente sencillos pero necesarios para el desarrollo de este trabajo. He encontrado natural el dividirlo en dos secciones: una de topología y otra de análisis, ya que son las principales ramas en las que se fundamenta el contenido de esta memoria. En la sección de topología, la mayor parte de las definiciones y primeras proposiciones han sido obtenidas de los portales web mencionados en la sección 4.2, aunque también se pueden consultar en los apuntes de la asignatura de *Topología de Espacios Métricos* [Alías Linares, 2017]. Estos desembocan en un resultado bien conocido, el hecho de que una aplicación continua e inyectiva de un espacio topológico compacto en un Hausdorff es un homeomorfismo, que será crucial en futuras demostraciones de secciones y capítulos próximos. También se incluye un apartado con algunas cuestiones relativas a la topología producto, entre las cuales se encuentra el teorema de Tychonoff. Estas serán fundamentales para hallar una topología para el espacio de Cantor visto como producto de espacios topológicos tras su identificación como el conjunto Δ, que será el protagonista del último subapartado de topología. Este espacio cuenta con unas propiedades que serán elaboradas y más tarde empleadas en la búsqueda del espacio de Banach separable universal inyectivo del Capítulo 3. Respecto a la sección relativa a análisis, en ella tan solo se definen las ideas más esenciales de los espacios de Banach, de las cuales cabe destacar las de topología débil y topología débil estrella. Por último, se incluye un corolario basado en el teorema de Hahn-Banach, otro resultado útil para la demostración del teorema de Banach-Mazur, que se puede encontrar en el Capítulo 3.

El siguiente es el **Capítulo 2**, que consiste en una extensa introducción a los retículos de Banach inspirada en la tesis de José David Rodríguez Abellán [Rodríguez Abellán, 2021] y en las notas de la charla que A. W. Wickstead dio en la Positivity V conference de julio de 2007 en Belfast [Wickstead, 2007]. En primer lugar, se expone el concepto de retículo vectorial, un espacio vectorial real ordenado cerrado para ínfimos y supremos en el cual las operaciones de suma y producto garantizan la compatibilidad entre el orden y la estructura de espacio vectorial. Esto induce a la definición de retículo de Banach, un retículo vectorial que además cuenta con una norma compatible con el orden y con la cual es un espacio de Banach. Esta primera subsección está acompañada de gran cantidad de ejemplos extraídos de los libros [Meyer-Nieberg, 1991] y [Schaefer, 1974], además de propiedades elementales de los retículos vectoriales y una serie de desigualdades e identidades que serán necesarias en el Capítulo 4. Cabe destacar el caso de las funciones reales continuas $C(K)$ definidas en un cierto conjunto compacto K con la norma del infinito y el orden punto a punto o el caso de las funciones $L_p(\mu)$ con su respectiva norma $\|\cdot\|_p$ y el orden en casi todo punto como ejemplos fundamentales de retículos de Banach. Los dos siguientes subapartados de este capítulo se centran en cuestiones derivadas de las definiciones previas que serán aplicadas en el propio análisis del artículo en el capítulo final. Por un lado, se deduce el hecho de que todo retículo de Banach es arquimedeano, es decir, cumple la propiedad arquimedeana. Por otro lado, los conceptos de ideales y bandas en el contexto de un retículo vectorial, en especial de bandas de proyección, llevan al teorema espectral de Freudenthal y en particular a un importante corolario fruto de varias de las proposiciones de esta sección que también será clave para el principal resultado de esta memoria. Finalmente, resulta interesante conocer una versión del teorema de Hahn-Banach para operadores positivos aplicada a retículos de Banach, cuya demostración es completamente análoga a la original.

La finalidad del **Capítulo 3** es la de encontrar un espacio de Banach que cumpla con el objetivo de esta memoria, es decir, que contenga una copia isométrica de todos los espacios de Banach separables. El teorema de Banach-Mazur no solo demuestra este hecho, sino que prueba que dicho espacio de Banach es el espacio de las funciones reales continuas $C[0,1]$

definidas en el intervalo $[0,1]$ o, en su defecto, también se puede argumentar la validez del conjunto Δ mediante la demostración expuesta. Para poder llegar a este resultado, es imprescindible apoyarse en el teorema de Banach-Alaoglu, el cual afirma que la bola cerrada del dual de un espacio de Banach dotado de la topología débil estrella relativa es compacta. Una aplicación práctica del teorema es comparar mediante el orden dado por la inclusión los conjuntos $C[0,1]$ y $L_1[0,1]$. Es evidente que viéndolos como espacios de Banach serán comparables y, más precisamente, $L_1[0,1]$ tendrá una copia isométrica dentro de $C[0,1]$ a consecuencia del teorema de Banach-Mazur. En cambio, si los pretendemos comparar como retículos de Banach, es decir, si buscamos un homomorfismo inyectivo de retículos de Banach entre ellos, veremos que es imposible mediante un sencillo procedimiento por reducción al absurdo. Esto será destacable en el siguiente capítulo, ya que no es casualidad que el retículo de Banach universal inyectivo para la clase de los retículos de Banach separables, es decir, aquel que contiene una copia isométrica de todos los demás, sea una mezcla de ambos.

El principal propósito de esta memoria, ya comentado anteriormente, se expone en el **Capítulo 4**. Considero humildemente que la mayor parte de mi contribución a este problema se ve reflejada aquí, ya que a pesar de exponer una prueba obtenida del artículo *Separable Universal Banach Lattices* [Leung et al., 2019], he sido capaz de esquematizar y detallar todas y cada una de las ideas en él presentes. Las gráficas, lemas y resultados previos a la demostración del resultado que he creado en este capítulo facilitan y simplifican su entendimiento, aunque a lo largo de toda la memoria también se pueden apreciar muchos otros detalles fundamentales que estaban ausentes y no aparecían en el ya citado artículo, por ejemplo, algunas identidades que se cumplen en retículos vectoriales, las definiciones y propiedades de las bandas e ideales o el teorema espectral de Freudenthal, todos ellos presentes en el Capítulo 2. Para la prueba del resultado se presenta un nuevo concepto, el de árbol finitamente ramificado, que no es más que un árbol en cuyos vértices se sitúa un vector positivo de un retículo de Banach que es igual a la suma de los vectores disjuntos dos a dos que se asignan a todos los vértices que salen de él; y se recurre repetidamente a la evidente observación de que cada nivel del árbol compone una partición del vector del vértice raíz. La conclusión del teorema no es solo la existencia de un retículo de Banach universal inyectivo para la clase de los retículos de Banach separables, sino que muestra que se trata del retículo de las funciones continuas de Δ en las funciones $L_1[0,1]$.

Las fuentes para la redacción de esta memoria están recogidas al final de ella, aunque se irán citando a lo largo de todo el trabajo en las introducciones de las distintas secciones. Si bien es cierto que existe una bibliografía en la que aparecen todos los libros y artículos consultados, he creado un capítulo previo en el que aparece un listado extenso de las páginas web visitadas que han contribuido a su elaboración, principalmente del capítulo introductorio.

Abstract

This abstract is intended to serve as an introduction to the content of this memoir, so as to give the reader a general idea of it. Although it is difficult to summarize something so extensive, its most important concepts will be mentioned in order to illustrate each of the chapters that constitute it as much as possible, making a brief journey through each one of them. Since the study of Banach lattices is not included in the program of the Bachelor of Science in Mathematics at Universidad de Murcia, this report starts from a basis that is assumed to have been acquired in courses such as *Topology of Metric Spaces* or *Functional Analysis* and, from there, all the other necessary notions are defined. Despite this, it seems coherent and logical to review all those basic criteria that are supposed to be already known in a first introductory chapter to facilitate its understanding.

The main objective of this memoir is to expose a separable Banach lattice for which any other can be identified as an isometric copy of it, that is, a universal injective Banach lattice for the class of all separable Banach lattices. This result, detailed in Chapter 4 through an in-depth analysis of the proof that can be found in the article *Separable Universal Banach Lattices* [Leung et al., 2019], is fundamental because it allows us to be able to compare any Banach lattice to a particular one by the order given by inclusion. Before facing the proof of the aforementioned article, a contextualization in concepts such as vector lattice and Banach lattice, and a series of essential properties needed for their demonstration are presented. In addition, I have considered of great interest to first show an equivalent version for Banach spaces, the Banach-Mazur theorem, which raises the need to develop it more specifically for Banach lattices.

Through **Chapter 1** an introduction to reasonably simple but necessary concepts for the development of this work is made. I have found it natural to divide it into two sections: one for topology and the other for analysis, since they are the main branches on which the content of this memoir is based. In the topology section, most of the definitions and first propositions have been obtained from the web portals mentioned in the 4.2 section, although they can also be consulted in the notes of the course of *Topology of Metric Spaces* [Alías Linares, 2017]. All these lead to a well-known result, the fact that a continuous injective application from a compact topological space into a Hausdorff space is a homeomorphism, which will be crucial in future proofs of the upcoming sections and

chapters. It also includes a section with some questions related to product topology, among which is Tychonoff's theorem. These will be fundamental to find a topology for the Cantor space seen as a product of topological spaces after its identification as the set Δ, which will be the focus of our last topology subsection. This space has some properties that will be developed and later used in our search for a separable universal injective Banach space in Chapter 3. Regarding the section on analysis, it only defines the most essential ideas of Banach spaces, of which those of weak topology and weak star topology should be noted. Finally, a corollary based on the Hahn-Banach theorem is included, another useful result for the proof of the Banach-Mazur theorem, which can be found in Chapter 3.

Next is **Chapter 2**, which consists of an extensive introduction to Banach lattices inspired by the thesis of José David Rodríguez Abellán [Rodríguez Abellán, 2021] and by the notes from the talk that AW Wickstead gave at the Positivity V conference July 2007 in Belfast [Wickstead, 2007]. In the first place, the concept of vector lattice is exposed, a real ordered vector space closed for infima and suprema in which the operations of addition and product guarantee the compatibility between the order and the vector space structure. This leads to the definition of a Banach lattice, a vector lattice that has also a norm compatible with the order and with which it is a Banach space. This first subsection is accompanied by a large number of examples taken from the books [Meyer-Nieberg, 1991] and [Schaefer, 1974], as well as elementary properties of vector lattices and a series of inequalities and identities that will be necessary in Chapter 4. It is worth highlighting the case of the continuous real functions $C(K)$ defined in a certain compact set K with the uniform norm and the pointwise order or the case of the functions $L_p(\mu)$ with their respective norm $\|\cdot\|_p$ and the order in almost every point as fundamental examples of Banach lattices. The next two subsections of this chapter focus on issues derived from the previous definitions that will be applied in the analysis of the article itself in the final chapter. On the one hand, the fact that every Banach lattice is Archimedean, that is, that fulfills the Archimedean property, follows. On the other hand, the concepts of ideals and bands in the context of a vector lattice, especially projection bands, lead to Freudenthal's spectral theorem and, in particular, to an important corollary resulting from several of the propositions in this section that will also be basic for the main result of this memoir. Finally, it is interesting to know a version of the Hahn-Banach theorem for positive operators applied to Banach lattices, whose proof is completely analogous to the original one.

The purpose of **Chapter 3** is to find a Banach space that meets the objective of this report, that is, that contains an isometric copy of all separable Banach spaces. The Banach-Mazur theorem not only proves this fact, but also proves that this Banach space is the space of continuous real functions $C[0,1]$ defined on the interval $[0,1]$ or, alternatively, the validity of the set Δ can also be argued by the exposed proof. In order to reach this result, it is essential to rely on the Banach-Alaoglu theorem, which states that the closed ball of the dual of a Banach space endowed with their weak star relative topology is compact. A practical application of the theorem is to compare the sets $C[0,1]$ and $L_1[0,1]$ by means of the order given by inclusion. It is obvious that by regarding them as Banach spaces they will be comparable and, more precisely, $L_1[0,1]$ will have an isometric copy within $C[0,1]$ as a consequence of the Banach-Mazur theorem. On the other hand, if we try to compare them as Banach lattices, that is, if we look for an injective homomorphism of Banach lattices between them, we will see that this is impossible by making use of a simple procedure of reducing it to an absurdity. This will be emphasized in next chapter, since it is not a coincidence that the universal injective Banach lattice for the class of all separable Banach lattices, that is, the one that contains an isometric copy of all others, is a mixture of both of them.

The main purpose of this memoir, already commented previously, is exposed in **Chapter 4**. I humbly consider that my major contribution to this problem is reflected here, since despite of exposing a proof obtained from the article *Separable Universal Banach Lattices* [Leung et al., 2019], I have been able to outline and detail each and every idea presented in it. The graphics, lemmas and results prior to the demonstration of the result that I have created in this chapter facilitate and simplify its understanding, although throughout this report many other fundamental details that were absent and did not appear in the aforementioned article can be found, for example, some identities that are fulfilled in vector lattices, the definitions and properties of bands and ideals, or Freudenthal's spectral theorem, all of them present in Chapter 2. To proof the result a new concept is presented, that of finitely branched tree, which is basically a tree at whose vertices a positive vector of a Banach lattice that is equal to the sum of the pairwise disjoint vectors that are assigned to all vertices that come out of it is placed; and the obvious observation that each level of the tree composes a partition of the vector of the root vertex is highly recurrent. The conclusion of the theorem is not only the existence of a universal injective Banach lattice for the class of all separable Banach lattices, but also shows that it is the lattice of the continuous functions from Δ into the functions $L_1[0,1]$.

The resources for this memoir are listed at the end of it, although they will be cited throughout the whole report in the introductions of its different sections. While it is true that there is a bibliography in which all the books and articles consulted appear, I have created a previous chapter where an extensive list of all the visited webpages that have contributed to its elaboration, mainly to the introductory chapter, can be found.

CHAPTER 1

Introduction

1.1 Topology

In this section a great deal of useful and well-known topological results will be introduced. Despite the fact that some of them are very elementary, they will serve as a basis for proofs of complex ideas described in the following chapters, specially Chapter 3. These were extracted from [Kechris, 1995] and from the links that can be found in Section 4.2.

1.1.1 Basic definitions and properties

Definition 1.1. Let (X, d) be a metric space and (X, τ_d) be the topological space induced by d. Then for any topological space which is homeomorphic to such a (X, τ_d), we say it is *metrizable*.

Definition 1.2. Let $T = (T, \tau)$ be a topological space and $H \subseteq T$ be a subset. Then H is *dense in* T if and only if the intersection of H with every open subset of T is non-empty.

Definition 1.3. A topological space $T = (T, \tau)$ is *separable* if and only if there exists a countable subset of T which is dense in T.

Definition 1.4. Let $T = (T, \tau)$ be a topological space. Then $\mathcal{C}$ is an *open cover for* T if $\mathcal{C} \subset \tau$ and $T \subseteq \cup\mathcal{C}$ where $\cup\mathcal{C}$ denotes the union of all elements of $\mathcal{C}$. In this case we say that *T is covered by $\mathcal{C}$*. Then T is *compact* if and only if every open cover for T has a finite subcover, i.e. a finite set which is also a cover for T.

Proposition 1.5. *Let T_1 and T_2 be topological spaces and let $f : T_1 \to T_2$ be a continuous mapping. If T_1 is compact then so is its image $f(T_1)$ under f. That is, compactness is a continuous invariant.*

Proof. Suppose $\mathcal{U}$ is an open cover of $f(T_1)$ by open sets in T_2. Because f is continuous, it follows that $f^{-1}(U)$ is open in T_1 for all $U \in \mathcal{U}$. The set $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of T_1, because for any $x \in T_1$, it follows that $f(x)$ must be in some $U \in \mathcal{U}$. Because T_1 is compact, it has a finite subcover $\{f^{-1}(U_1), f^{-1}(U_2), ..., f^{-1}(U_r)\}$. It follows that $\{U_1, U_2, ..., U_r\}$ is a finite subcover of $f(T_1)$. $\square$

Proposition 1.6. *Let T be a compact space and let C be a closed subspace of T. Then C is compact. That is, the property of being compact is weakly hereditary.*

Proof. Let $\mathcal{U}$ be an open cover of C. Since C is closed, it follows by definition of closed that $T \backslash C$ is open in T. So if we add $T \backslash C$ to $\mathcal{U}$, we see that $\mathcal{U} \cup T \backslash C$ is also an open cover of T. As T is compact, there exists a finite subcover of $\mathcal{U} \cup T \backslash C$, say $V = U_1, U_2, ..., U_r$. This covers C by the fact that it covers T. If $T \backslash C$ is an element of V, then it can be removed from V and the rest of V still covers C. Thus, we have a finite subcover of $\mathcal{U}$ which covers C, and hence C is compact. $\square$

Proposition 1.7. *Let (X, d) be a metric space. If (X, τ_d) [1] is compact, then it is separable.*

Proof. First, we state that given $\epsilon > 0$ the set $\{B(x, \epsilon) : x \in X\}$ is an open cover for (X, τ_d). This is true as the open balls form a basis for the topology of X induced by its metric and it is clear that $X \subseteq \bigcup\{B(x, \epsilon) : x \in X\}$. Now as X is compact, the given cover has a finite subcover of the desired form. So there exists a finite set F_ϵ such that $\{B(x, \epsilon) : x \in F_\epsilon\}$ is an open cover of X. This means that for any $y \in X$, there exists a $x \in F_\epsilon$ such that $d(x, y) < \epsilon$. Now repeating the process for each $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$ we can define the set $F := \cup\{F_{\frac{1}{n}} : n \in \mathbb{N}\}$ which is countable because of being a countable union of finite sets. Lastly we see that F is a dense set by proving that the intersection of F with every open set O of X is non-empty: for each $y \in O$, since O is open, there exists an integer N such that $B(y, \frac{1}{N}) \subset O$. Now as the collection $\{B(x, \frac{1}{N}) : x \in F_{\frac{1}{N}}\}$ is a finite cover for X, there is some $x \in F_{\frac{1}{N}}$ such that $d(x, y) < \frac{1}{N}$. But then trivially $x \in B(y, \frac{1}{N}) \subset O$, and so $O \cap F$ is non-empty. This completes the proof of the claim. $\square$

Definition 1.8. Let T_1 and T_2 be topological spaces. A mapping $f : T_1 \to T_2$ is a *topological embedding* of T_1 into T_2 if f is an homeomorphism onto its image $f(T_1)$. In this case we say that T_1 *embeds homeomorphically into* T_2. We can also state that T_2 *contains a homeomorphic copy of* T_1 (specifically, $f(T_1)$) or that T_1 *is homeomorphic to a topological subspace of* T_2.

Proposition 1.9. *Let T_1 be a compact space and T_2 a topological space. If there exists some homeomorphism $f \colon T_1 \to T_2$, then T_2 is a compact space. That, is compactness is a topological property.*

Proof. Let $\{O_i\}_{i \in I}$ be an open cover for T_2. By the continuity of f, the preimages form an open cover $\{f^{-1}(O_i)\}_{i \in I}$ for T_1. Hence by compactness of T_1, there exists a finite subset $F \subset I$ such that $\{f^{-1}(O_i)\}_{i \in F}$ is still an open cover for T_1. Finally, by surjectivity of f it follows that

$$T_2 = f(T_1) = f(\cup_{i \in F} f^{-1}(O_i)) = \cup_{i \in F} O_i$$

where we used that images of unions are unions of images. This means that also $\{O_i\}_{i \in F}$ is still an open cover for T_2, and in particular a finite subcover of the original cover. $\square$

[1] See Definition 1.1

Definition 1.10. Let $T = (T, \tau)$ be a topological space. T is a *Hausdorff space* or T_2 *space* if and only if for any two distinct elements $x, y \in T$ there exist disjoint open sets $U, V \in \tau$ containing x and y respectively.

Proposition 1.11. *Let $T = (T, \tau)$ be a Hausdorff topological space and let $T_H = (T_H, \tau_H)$ be a subspace of T. Then T_H is a Hausdorff space. That is, the property of being a Hausdorff space is hereditary.*

Proof. By the definition of topological subspace, we have that $\emptyset \subset T_H \subseteq T$ and that $\tau_H := \{U \cap T_H : U \in \tau\}$. Now let $x, y \in T_H$ such that $x \neq y$. Then as $x, y \in T$ we have that there exist $U, V \in \tau$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. As $x, y \in T_H$ we have that $x \in U \cap T_H, y \in V \cap T_H$ with $(U \cap T_H) \cap (V \cap T_H) = \emptyset$ and so the Hausdorff property is satisfied in T_H. $\square$

Proposition 1.12. *Let $H = (H, \tau)$ be a Hausdorff topological space and let C be a compact subspace of H. Then C is closed in H.*

Proof. Let $a \in H \backslash C$. We are going to prove that there exists an open set U_a such that $a \in U_a \subseteq H \backslash C$. For any single point $x \in C$, the Hausdorff condition ensures the existence of disjoint open sets $U(x)$ and $V(x)$ containing a and x respectively. Suppose there were only a finite number of points $x_1, x_2, ..., x_r$ in C. Then we could take $U_a = \bigcap_{i=1}^{r} U(x_i)$ and get $a \in U_a \subseteq H \backslash C$. Now suppose C is not finite. The set $\{V(x) : x \in C\}$ is an open cover for C. As C is compact, it has a finite subcover, say $\{V(x_1), V(x_2), ..., V(x_r)\}$. Let $U_a = \bigcap_{i=1}^{r} U(x_i)$. Then U_a is open because it is a finite intersection of open sets. Also, $a \in U_a$ because $a \in U(x_i)$ for each $i = 1, 2, ..., r$. Finally, if $b \in U_a$ then for any $i = 1, 2, ..., r$ we have $b \in U(x_i)$. Because $C \subseteq \bigcup_{i=1}^{r} V(x_i)$ with $b \notin V(x_i)$, then $b \notin C$. Thus: $U_a \subseteq H \backslash C$. Then $H \backslash C$ is open and it follows that C is closed. $\square$

Lemma 1.13. *Let (X, d) be a metric space. Then $\hat{d}(x, y) := \min\{d(x, y), 1\}$ is an equivalent metric for X. That is, the topologies induced by d and $\hat{d}$ on X are the same.*

Proof. First we see that for any arbitrary element $x \in X$ then given $\epsilon > 0$ there exists some $\delta > 0$ such that $B_d(x, \delta) \subseteq B_{\hat{d}}(x, \epsilon)$. By taking $\delta := \epsilon$, we have that $y \in B_d(x, \delta)$ means that $d(x, y) < \epsilon$. Then, as $\hat{d}(x, y) \leq d(x, y)$ we have $y \in B_{\hat{d}}(x, \epsilon)$.

Reciprocally, we will prove that for any arbitrary element $x \in X$ then given $\epsilon > 0$ there exists some $\delta > 0$ such that $B_{\hat{d}}(x, \delta) \subseteq B_d(x, \epsilon)$. If $\epsilon \leq 1$, then by choosing $\delta := \epsilon$, we have that $y \in B_{\hat{d}}(x, \delta)$ means that $\hat{d}(x, y) < \epsilon$. Now by definition must be $\hat{d}(x, y) = d(x, y)$ and consequently $y \in B_d(x, \epsilon)$. On the other hand, if $\epsilon > 1$ then we take $\delta := 1$ so that $y \in B_{\hat{d}}(x, \delta)$ means $\hat{d}(x, y) < 1$ so again by definition must be $\hat{d}(x, y) = d(x, y)$. Therefore $d(x, y) < 1 \leq \epsilon$ and $y \in B_d(x, \epsilon)$. $\square$

1.1.2 The product topology

Definition 1.14. Let $\{X_i : i \in I\}$ be an indexed family of topological spaces where I is an arbitrary index set and let X be its cartesian product: $X := \prod_{i\in I} X_i$. The *product topology*, sometimes called the *Tychonoff topology*, on X is the topology that has as a basis the set of all $\prod_{i\in I} \mathcal{U}_i$ where $\mathcal{U}_i \subset X_i$ is open for every $i \in I$ and the set $\{i \in I : \mathcal{U}_i \neq X_i\}$ is finite.
The cartesian product X endowed with the product topology is called the *product space*.

Lemma 1.15. *Let $\{X_i : i \in I\}$ be an indexed family of Hausdorff topological spaces where I is an arbitrary index set and let $X := \prod_{i\in I} X_i$ be its product space. Then X is a Hausdorff space.*

Proof. If x and y are distinct points of X, then there is at least one $i_0 \in I$ on which they differ, meaning that $x_{i_0} \neq y_{i_0}$. X_{i_0} is Hausdorff, so there are open sets U_{i_0} and V_{i_0} in X_{i_0} such that $x \in U_{i_0}$, $y \in V_{i_0}$, and $U_{i_0} \cap V_{i_0} = \emptyset$. Now let $U_i = V_i = X_i$ for each $i \in I \backslash \{i_0\}$, let $U = \prod_{i\in I} U_i$, and let $V = \prod_{i\in I} V_i$; then U and V are basic open sets in X, $x \in U$, $y \in V$, and $U \cap V = \emptyset$ (to see this, note that if $z \in U \cap V$, then it would be $z_{i_0} \in U_{i_0} \cap V_{i_0} = \emptyset$, so no such z can exist). Thus, X is Hausdorff. $\square$

Theorem 1.16 (Tychonoff's Theorem). *Let $\{(X_i, \tau_i) : i \in I\}$ be an indexed family of compact topological spaces where I is an arbitrary index set and let us consider its product space X. Then X is compact. That is, the arbitrary product of compact spaces is compact.*

Proof. In [Wildman, 2010], an interesting proof of the theorem using Alexander Subbase theorem can be found. $\square$

Proposition 1.17. *Let $\{(X_n, d_n) : n \in \mathbb{N}\}$ be a countable collection of metric spaces and let us consider its product space X as aforementioned. Define*

$$\begin{aligned} d\colon X \times X &\to \mathbb{R} \\ (x, y) &\mapsto \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n} \end{aligned}$$

for every $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$ in X. Then d is a metric on X whose induced topology is equivalent to the product topology on X.
That is, the countable product of metric spaces is metrizable.

Proof. First, let O be a basic open product set, so $O = \prod_n O_n$, all O_n are open in X_n and where we have a finite set $F \subset \mathbb{N}$ such that $n \notin F$ if and only if $X_n = O_n$. We want to show that it is open in the d-topology (topology induced by the metric d), so pick $x \in O$, and let us find $r > 0$ such that $B_d(x, r) \subset O$. This would prove that all basic product open sets are d-open, and thus all product open sets are d-open.

Now, for every $n \in F$, we have that $x_n \in O_n$, which is a (non-trivial) open subset in X_n, so we have some $r_n > 0$ such that $B_{d_n}(x_n, r_n) \subset O_n$, from the fact that the topology on X_n is induced by the metric d_n. As we have finitely many r_n to consider, we can find some $0 < r < 1$ such that $r \leq \frac{r_n}{2^n}$ for all $n \in F$.

The claim now is that $B_d(x,r) \subset O$. To see this, take any $y \in B_d(x,r)$ with $d(x,y) < r$. For $n \in F$, we know that $\frac{d_n(x_n,y_n)}{2^n} \leq d(x,y) < r \leq \frac{r_n}{2^n}$, which implies that for such n we have that $d_n(x_n,y_n) < r_n$, and so $y_n \in B_{d_n}(x_n,r_n) \subset O_n$. Hence, for all $n \in F$, $y_n \in O_n$, and as the rest of O_ns equal X_n by the definition of O, we have that indeed $y \in O$, and as y was arbitrary, $B_d(x,r) \subset O$, as required.

We now start with an open ball $B_d(x,r)$, a basic open subset of the d-topology, for some arbitrary $x \in X$ and $r > 0$, and try to find a basic open subset O in the product topology such that $x \in O \subset B_d(x,r)$. This would then show that any d-open ball is open in the product topology and would prove the other inclusion we need: every d-open set is product open. The intuition is that the tail of a series like the one that defines d is essentially irrelevant (we can get it as small as we like) and this corresponds to the idea that basic open subsets only depend on finitely many non-trivial open sets. So we first pick $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{r}{2}$ where N defines our tail. For $1 \leq k \leq N$ we consider the open balls $O_k = B_{d_k}(x_k, \frac{r}{2N})$, and we set $O_k = X_k$ for $k \geq N+1$.

The claim now is that $O = \prod_k O_k \subset B_d(x,r)$, as required. Note that O is indeed a basic open subset in the product topology on X and $x \in O$.

To verify the latter claim, we simply estimate: let y be in O, then for $k \leq N$, $d_k(x_k,y_k) < \frac{r}{2N}$, so

$$\sum_{k=1}^{N} \frac{d_k(x_k,y_k)}{2^k} \leq \sum_{k=1}^{N} d_k(x_k,y_k) < N \cdot \frac{r}{2N} = \frac{r}{2},$$

while

$$\sum_{k=N+1}^{\infty} \frac{d_k(x_k,y_k)}{2^k} \leq \sum_{k=N+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^N} < \frac{r}{2}.^2$$

Putting it together, we indeed get that for $y \in O$ we have $d(x,y) < \frac{r}{2} + \frac{r}{2} = r$, as required. □

1.1.3 A result on topological embeddings

Proposition 1.18. *Let $T_1 = (T_1, \tau_1)$ and $T_2 = (T_2, \tau_2)$ be topological spaces and let $f \colon T_1 \to T_2$ be a bijection. Then f is open if and only if f^{-1} is continuous.*

Proof. First, note that $g := f^{-1}$ is a bijection with $g^{-1} = f$. Now let f be open. Then, by definition of an open mapping, for all $H \in \tau_1, f(H) \in \tau_2$. But $f = g^{-1}$ so for all $H \in \tau_1, g^{-1}(H) \in \tau_2$ which is exactly the definition for g to be continuous.

The argument works the other way: let now g be continuous. Then by definition of continuous mapping for all $H \in \tau_1, g^{-1}(H) \in \tau_2$. But $g^{-1} = f$ so for all $H \in \tau_1, f(H) \in \tau_2$ which is exactly the definition for f to be open. □

Proposition 1.19. *Let $T_1 = (T_1, \tau_1)$ and $T_2 = (T_2, \tau_2)$ be topological spaces and let $f \colon T_1 \to T_2$ be a bijection. Then f is open if and only if f is closed.*

Proof. Suppose f is an open mapping. From the definition of open mapping, for all $H \in \tau_1, f(H) \in \tau_2$. As f is a bijection, $f(T_1 \backslash H) = f(T_1) \backslash f(H) = T_2 \backslash f(H)$. By definition of closed set, $T_1 \backslash H$ is closed in T_1 and, as f is an open mapping, $f(T_1 \backslash H) = T_2 \backslash f(H)$ is closed in T_2. Hence by definition f is a closed mapping. The analogous argument proves that if f is closed then it is open. $\square$

Corollary 1.20. *Let $T_1 = (T_1, \tau_1)$ and $T_2 = (T_2, \tau_2)$ be topological spaces and let $f : T_1 \to T_2$ be a bijection. Then f is closed if and only if f^{-1} is continuous.*

Proof. This is an elementary consequence of Propositions 1.18 and 1.19. $\square$

Theorem 1.21. *Let T_1 be a compact space, T_2 be a Hausdorff space and let $f : T_1 \to T_2$ be a continuous injection. Then f determines a homeomorphism from T_1 to $f(T_1)$. That is, f is a topological embedding of T_1 into T_2.*

Proof. We first show that $f : T_1 \to T_2$ is closed. We are to show that if V is closed in T_1, then $f(V)$ is closed in T_2. Suppose V is closed in T_1. Then, since T_1 is compact, we know V is compact (see Proposition 1.6). So $f(V)$ is compact from Proposition 1.5. Since T_2 is Hausdorff, $f(V)$ closed by Proposition 1.12. Thus, applying Corollary 1.20, it follows that f is an homeomorphism onto its image. $\square$

Corollary 1.22. *If K is a compact set for some topology τ, and τ' is any Hausdorff topology on K which is weaker than τ (that is, $\tau' \subseteq \tau$), then τ and τ' coincide.*

Proof. Suppose A is a τ-closed subset of K. We know that the mapping $id_K : (K, \tau) \to (K, \tau')$ is a trivial continuous injection. Then, by Theorem 1.6 and Corollary 1.20, A is τ'-closed. This means $\tau \subseteq \tau'$ and consequently both topologies coincide. $\square$

1.1.4 The Cantor set

Definition 1.23. The *Cantor set* $\mathcal{C}$ consists of all those real numbers x in $[0, 1]$ so that when we write x in ternary form $x = \sum_{i=1}^{\infty} a_i/3^i$, then none of the numbers $a_1, a_2, ...$ equals 1 (i.e., either $a_i = 0$ or $a_i = 2$).
As $\mathcal{C}$ is a topological subspace of $\mathbb{R}$, the topology of the Cantor set is its relative topology, that is, the one induced by the usual topology on $\mathcal{C}$.

Proposition 1.24. *The Cantor set*[3] *$\mathcal{C}$ is homeomorphic to $\Delta := \{0, 1\}^{\mathbb{N}}$.*

Proof. Let us establish a homeomorphism

$$\begin{aligned} f : \{0,1\}^{\mathbb{N}} &\to \mathcal{C} \\ (x_n)_{n=1}^{\infty} &\mapsto \sum_{n=1}^{\infty} \frac{2x_n}{3^n} \end{aligned}$$

First, since it has a converging geometric series as a majorant, the series converges and even uniformly. By the definition of Cantor set given in the first chapter, it is clear that the image lives in $\mathcal{C}$.

[3]From now on, we refer to $\mathcal{C}$ as the $Cantor\ middle\ third\ set$ and the Cantor set will be regarded as Δ.

From Definition 1.23, $\mathcal{C}$ is a metrizable space for the usual metric in $\mathbb{R}$. Now as the set $\{0,1\}$ is a metric space for the discrete metric ($d(x,y)=1$ if $x\neq y$ and $d(x,y)=0$ otherwise), by Proposition 1.17 then $\{0,1\}^{\mathbb{N}}$ is also a metric space for $d(x,y)=\sum_{n=1}^{\infty}\frac{|x_n-y_n|}{2^n}$ for every $x=(x_n)_{n=1}^{\infty}$ and $y=(y_n)_{n=1}^{\infty}$.

A mapping between metric spaces $f\colon (X,d_X)\to(Y,d_Y)$ is continuous if for every $\epsilon>0$ there exists $\delta>0$ such that $d_Y(f(x),f(y))<\epsilon$ whenever $d_X(x,y)<\delta$. Now given some $\epsilon>0$, if we choose $\delta:=\epsilon$ we obtain

$$|\sum_{n=1}^{\infty}\frac{2x_n}{3^n}-\sum_{n=1}^{\infty}\frac{2y_n}{3^n}|=|\sum_{n=1}^{\infty}\frac{2(x_n-y_n)}{3^n}|\leq\sum_{n=1}^{\infty}\frac{2|x_n-y_n|}{3^n}<2\cdot\epsilon\cdot\sum_{n=1}^{\infty}\frac{1}{3^n}=2\cdot\epsilon\cdot\frac{1}{2}=\epsilon$$

which proves continuity. Injectivity can be concluded from the fact that $|f(x)-f(y)|>0$ if $x\neq y$. Surjectivity is straightforward from the definition since for every element of the Cantor set it can be given such a series by choosing every x_i wisely ($\forall y\in\mathcal{C}$, choose $x_i=0$ if at the i-th step y is located on the left part of the i-th step interval being split to thirds, and $x_i=1$ if on the right).

Lastly, recalling Proposition 1.11 and the fact that $\mathbb{R}$ is Hausdorff, $\mathcal{C}$ is Hausdorff and by Theorem 1.16 and the fact that $\{0,1\}$ is compact (note that every finite set is trivially compact) then $\{0,1\}^{\mathbb{N}}$ must be compact. As a consequence from Corollary 1.21, we conclude that f is a homeomorphism. □

Proposition 1.25. *The Cantor set Δ can be homeomorphically embedded into the real interval* $[0,1]$.

Proof. Let us use the homeomorphism $f\colon\{0,1\}^{\mathbb{N}}\to\mathcal{C}\subseteq[0,1]$ which we know is injective (see Proposition 1.24). □

The following proof was extracted from [Simón Pinero, 2017] in order to be used below.

Lemma 1.26. *The cartesian product $\mathbb{N}\times\mathbb{N}$ is countable.*

Proof. For this we will use the fact that given two sets A,B then $|A|\leq|B|$ if there exists some injective mapping $f\colon A\to B$.
First, we define

$$\begin{aligned} f\colon \mathbb{N} &\longrightarrow \mathbb{N}\times\mathbb{N}\\ n &\longmapsto (n,0)\end{aligned}$$

which is trivially injective. Thus, $|\mathbb{N}|\leq|\mathbb{N}\times\mathbb{N}|$.

Secondly, let us display an injective mapping $\varphi:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$. To simplify the arguments we will assume $\mathbb{N}=\{1,2,3,...\}$, without 0. The idea is to order the pairs in the lexicographic order and then count them "diagonally". Note that in each diagonal from $(1,n)$ to $(n,1)$ there are exactly n pairs whose coordinates add up to $n+1$, and before reaching that diagonal $\sum_{i=1}^{n-1} i$ pairs have been already counted. Then when finishing the diagonal we will have counted

$$\sum_{i=1}^{n-1} i + n = \sum_{i=1}^{n} i$$

pairs. If we call $S(n)$ to the sum of the n first natural numbers, it is obvious that $(1, n) \mapsto S(n-1) + 1$, $(2, n-1) \mapsto S(n-1) + 2$ and thus $(n, 1) \mapsto S(n-1) + n = S(n)$.

This way, the correspondence remains as follows. If an arbitrary element (i, j) is considered, then it belongs to the diagonal of $(1, i+j-1) \mapsto S(i+j-2) + 1$. Then $(i, j) \mapsto S(i+j-2) + i$, and applying the widely known formula of the sum we obtain

$$\varphi(i, j) = \frac{(i+j-1)(i+j-2)}{2} + i.$$

Because of the way we have defined this mapping we know it will be injective. Hence, $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$ which finally gives $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$ and the fact that the cartesian product is countable. $\square$

Proposition 1.27. *There exists a continuous surjection from the Cantor set Δ into the real interval $[0, 1]^{\mathbb{N}}$. That is, $[0, 1]^{\mathbb{N}}$ is a continuous image of Δ.*

Proof. First let us define the continuous surjection

$$\begin{aligned} f\colon \{0,1\}^{\mathbb{N}} &\to [0,1] \\ (x_n)_{n=1}^{\infty} &\mapsto \sum_{n=1}^{\infty} \frac{x_n}{2^n} \end{aligned}$$

which maps Δ continuously onto $[0, 1]$. As in Proposition 1.24, we will prove continuity regarding the spaces as metric spaces with their metrics already described. Given some $\epsilon > 0$, if we choose $\delta := \epsilon$ we obtain

$$|\sum_{n=1}^{\infty} \frac{x_n}{2^n} - \sum_{n=1}^{\infty} \frac{y_n}{2^n}| = |\sum_{n=1}^{\infty} \frac{x_n - y_n}{2^n}| \leq \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n} < \epsilon \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} = \epsilon \cdot 1 = \epsilon$$

which proves that f is continuous. Naturally, the functions displayed is a surjection as the elements $x \in [0, 1]$ can be we written in their binary form $x = \sum_{n=1}^{\infty} a_n/2^n$.
Now recalling Lemma 1.26, we can state that $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}\times\mathbb{N}}$. Hence, combining both functions, the mapping $\hat{f}\colon \{0,1\}^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ given by $\hat{f}(x) := f(x)_{n=1}^{\infty}$ for every $x = (x_n)_{n=1}^{\infty}$ is a continuous surjection that fits the statement. $\square$

Definition 1.28. A closed set F in a topological space X is a *retract* of X if there is a continuous surjection $f\colon X \to F$ such that $f(x) = x$ for $x \in F$.

Lemma 1.29. *Let $(a_n)_{n=1}^{\infty}$ be a sequence in a space X and $a \in X$ such that for every subsequence of $(a_n)_{n=1}^{\infty}$ there is a subsequence that converges to a. Then $(a_n)_{n=1}^{\infty}$ converges to a.*

Proof. Suppose it does not converge to a. Then, we can find a neighbourhood of a and a subsequence of $(a_n)_{n=1}^{\infty}$ such that it never enters the neighbourhood. But this subsequence can have no subsequence converging to a, a contradiction. $\square$

Lemma 1.30. *Every non-empty closed subset C of the Cantor set Δ is a retract of it.*

Proof. Let us consider the metric $d(x,y) := \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{3^n}$ in Δ, regarding Δ as a subset of elements in ternary form of $\mathbb{R}$ this time. Note that if we have $d(x,y) = d(x,z)$ then it must be $|x-y| = |x-z|$ in the real line and hence $y = z$ or $x = \frac{y+z}{2}$. By construction of the Cantor set [4], we know that it is absurd that any $x \in \Delta$ is the midpoint of two points in the set. Thus, must be $y = z$.

Now for any fixed element $x \in \Delta$ we know that the function that assigns its distance to any element in C, $d(x,\cdot)\colon C \to \mathbb{R}$, is defined in a compact domain as C is closed in Δ (see Proposition 1.6) and is continuous by the triangle inequality, $|d(x,y) - d(x,z)| \le d(y,z)$, so it must attain a minimum in C.

Then we can define the function $r\colon \Delta \to C$ as $d(x, r(x)) = d(x, C) = \inf\{d(x,y) : y \in C\}$, that is, $r(x)$ is the closest point in C for any $x \in \Delta$, which exists and is unique by our previous observations.

Clearly, r is continuous. Let $x_n \mapsto x$ in Δ and let us take an arbitrary subsequence $r(x_{n_k})$ of $r(x_n)$. As C is compact, then it has a subsequence $r(x_{n_{k_i}})$ which is convergent to some $y \in C$. Lastly, as the function $d\colon \Delta \times \Delta \to \mathbb{R}$ is continuous by the triangle inequality, $|d(x_n, y_n) - d(x,y)| \le d(x_n, x) + d(y_n, y)$, and we know that $d(x_{n_{k_i}}, r(x_{n_{k_i}})) \le d(x_{n_{k_i}}, z)$ for any $z \in C$, taking limits we have that $d(x,y) \le d(x,z)$ and so it must be $r(x) = y$.

Hence, by Lemma 1.29, $r(x_n)$ converges to $r(x)$ and then r is a retract of the Cantor set onto C. $\square$

Proposition 1.31. *Every compact metrizable space X is homeomorphic to a compact*[5] *subset of $[0,1]^{\mathbb{N}}$.*

Proof. Being compact and metrizable, X contains a countable dense set, $(x_n)_{n=1}^{\infty}$, as it is separable by Proposition 1.7. Let d be a metric on K inducing its topology. Without loss of generality we can assume that $0 \le d \le 1$ (see Lemma 1.13). Now we define $f(x) = (d(x, x_n))_{n \in \mathbb{N}}$.

Clearly, f is continuous since every coordinate mapping $x \mapsto d(x, x_n)$ is continuous for each n. Check that given $\epsilon > 0$ we can take $\delta := \epsilon$ as whenever $d(x,y) < \delta$ then $|d(x,x_n) - d(y,x_n)| \le |d(x,y)| < \epsilon$ by using the triangular inequality.

Now we see that f must be injective because if x and y are two different points in X then there exists some x_n such that $d(x,x_n) \ne d(y,x_n)$ and, therefore, $f(x)$ and $f(y)$ will differ in the nth-coordinate. For this note that if $d(x,x_n) = d(y,x_n)$ for every natural number $n \in \mathbb{N}$, then it is $d(x,y) \le d(x,x_n) + d(y,x_n) = 2 \cdot d(x,x_n)$ and as the set $(x_n)_{n=1}^{\infty}$

[4]Although Cantor himself defined the set in a general, abstract way, the most common modern construction is the Cantor ternary set, built by removing the middle third of a line segment and then repeating the process with the remaining shorter segments.

[5]Note that must be compact because of Proposition 1.9.

is dense we can always find some x_n such that $d(x, x_n) < \epsilon$ for any given $\epsilon > 0$. Hence, making it tend to zero we would have $d(x, y) = 0$, but this can only happen if $x = y$.

Since X is compact and $[0,1]^{\mathbb{N}}$ is Hausdorff (by using Lemma 1.15, Proposition 1.11 and the facts that $\mathbb{R}$ is Hausdorff and $[0,1]$ is a subset of $\mathbb{R}$) we can apply Corollary 1.21 and hence f maps K homeomorphically into its image. □

Proposition 1.32. *For every nonempty compact metrizable space K, there exists a continuous surjection from the Cantor set Δ onto K. That is, K is a continuous image of Δ.*

Proof. Since every compact metrizable space K is homeomorphic to a compact subset $\bar{K}$ of $[0,1]^{\mathbb{N}}$ by Proposition 1.31, there must exist some homemorphism $h\colon K \to \bar{K} \subset [0,1]^{\mathbb{N}}$. Clearly, its inverse $h^{-1}\colon \bar{K} \to K$ is continuous and surjective.

Now we consider the continuous surjection $f\colon \Delta \to [0,1]^{\mathbb{N}}$ from Proposition 1.27 and define the set $F := \{x \in \Delta : f(x) \in \bar{K}\}$. Note that the restriction $f|_F$ is continuous and surjective on its image, which is precisely $\bar{K}$.

As F is closed in Δ because of being the preimage by a continuous mapping f of a closed set $\bar{K}$, by Lemma 1.30 there is a continuous surjection $r\colon \Delta \to F$.

The scheme of the composition that completes our proof can be seen in the following diagram.

$$\Delta \xrightarrow{\ r\ } F \xrightarrow{f|_F} \bar{K} \xrightarrow{h^{-1}} K$$

This is obviously a continuous surjection from the Cantor set Δ into K, as it is a composition of continuous surjections. □

1.2 Analysis

The following analytical definitions and results are needed for Chapter 3 and can be found in [Albiac and Kalton, 2016] and [Cascales et al., 2012].

1.2.1 Basic definitions and properties

Definition 1.33. Let X and Y be normed linear spaces. A linear operator $T : X \to Y$ is an *isometry* (or *isometric embedding*) if for all $x \in X$ we have that $\|T(x)\| = \|x\|$. In this case we say that X *embeds isometrically into* Y. We can also state that Y *contains an isometric copy of* X (specifically, $T(X)$) or that X *is isometric to a normed subspace of* Y.

Definition 1.34. Let X and Y be normed linear spaces. A linear operator $T : X \to Y$ is an *embedding* of X into Y if T is an isomorphism onto its image $T(X)$. In this case we say that X *embeds in* Y or that Y *contains an isomorphic copy of* X (specifically, $T(X)$).

Lemma 1.35. *Let X and Y be normed linear spaces. If $T : X \to Y$ is an isometry, then T is injective. Thus, T is an embedding of X into Y.*

Proof. Suppose that $x, y \in X$ and $T(x) = T(y)$, then $T(x) - T(y) = 0$. So as T is a linear operator $0 = \|T(x) - T(y)\| = \|T(x - y)\| = \|x - y\|$. But this happens if and only if $x - y = 0$, so $x = y$. Hence, T is injective and an isomorphism onto its image $T(X)$ so the result follows. $\square$

Definition 1.36. Let X be a normed vector space. The *(topological) dual space of* X, denoted as X^*, is the Banach space $L(X, \mathbb{R})$ where $L(X, \mathbb{R})$ consists of all linear operators from X to $\mathbb{R}$ with the operator norm defined as $\|T\|_{op} := \sup\{\|T(x)\| : \|x\| = 1\}$ for such an operator T.

Definition 1.37. Let X be a normed vector space and let X^* be its dual space. The *positive part* of a subset $S \subseteq X^*$ is the set $\{x^* \in S : x^*(x) \geq 0, \forall x \geq 0\}$.

Definition 1.38. Let X be a normed vector space. The *weak topology of* X, usually denoted w-topology or $\sigma(X, X^*)$-topology, is the weakest topology on X such that each $x^* \in X^*$ is continuous.

Note that a base of neighborhoods of $x_0 \in X$ is given by the sets of the form

$$V_\epsilon(x_0; x_1^*, ..., x_n^*) = \{x \in X : |x_i^*(x - x_0)| < \epsilon, i = 1, ..., n\},$$

where $\epsilon > 0$ and $\{x_1^*, ..., x_n^*\}$ is any finite subset of X^*.

One can also give an alternative description of the weak topology via the notion of convergence: take a sequence $(x_n)_{n\in\mathbb{N}}$ in X; we will say that $(x_n)_{n\in\mathbb{N}}$ converges weakly to $x_0 \in X$, and we write $x_n \xrightarrow{\text{w}} x_0$, if for each $x^* \in X^*, x^*(x_n) \to x^*(x_0)$.

Definition 1.39. Let X be a normed vector space and let X^* be its dual space. Let $j \colon X \to X^{**}$ be the *natural embedding of a Banach space in its second dual*, given by $j(x)(x^*) = x^*(x)$. As usual we identify X with $j(X) \subset X^{**}$.

Observation 1.40. Notice also that when we identify X with $j(X)$ and consider X as a subspace of X^{**} this is not simply an identification of sets; actually $(X, \sigma(X, X^*)) \to j(X, \sigma(X^{**}, X^*))$ is a linear homeomorphism.

Definition 1.41. Let X be a normed vector space and let X^* be its dual space. The *weak* topology of* X, usually denoted w^*-topology or $\sigma(X^*, X)$-topology, is the weakest topology on X^* by X, i.e. it is the weakest topology on X^* that makes all linear functionals in $X \subset X^{**}$ continuous.

Note that a base of neighborhoods of $x_0^* \in X^*$ is given by the sets of the form

$$W_\epsilon(x_0^*; x_1, ..., x_n) = \{x^* \in X^* : |x^*(x_i) - x_0^*(x_i)| < \epsilon, i = 1, ..., n\},$$

for any $\epsilon > 0$ and $\{x_1, ..., x_n\}$ any finite subset of X.

As before, we can equivalently describe the weak* topology of a dual space in terms of convergence: we say that a sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^* converges weakly* to $x_0^* \in X^*$, and we write $x_n^* \xrightarrow{\text{w}^*} x_0^*$, if for each $x \in X$, $x_n^*(x) \to x_0^*(x)$.

Of course, the weak* topology of X^* is no bigger than its weak topology and, in fact, $\sigma(X^*, X) = \sigma(X^*, X^{**})$ if and only if $j(X) = X^{**}$ (that is, if and only if X *is reflexive*).

1.2.2 Hahn-Banach theorem

Theorem 1.42 (Hahn-Banach theorem). *Let $(X, \|\cdot\|)$ a normed linear space over $\mathbb{R}$, $Y \subset X$ a linear subspace, and $y^*: Y \to \mathbb{R}$ a bounded linear operator ($y^* \in Y^*$). Then there exists some $x^*: X \to \mathbb{R}$ bounded linear operator ($x^* \in X^*$) such that:*

(i) (extension) $x^*(y) = y^*(y)\ \forall y \in Y$ ($x^*|_Y = y^*$)

(ii) (norm-preserving) $\|x^*\| = \|y^*\|$

In other words, the restriction operator $\phi\colon x^* \in X^* \mapsto x^*|_Y \in Y^*$ is suprajective and every $y^* \in Y^*$ is the restriction of some $x^* \in X^*$ that preserves the norm $\|x^*\| = \|y^*\|$.

Proof. See [Cascales et al., 2012, 3.1.7]. □

Corollary 1.43. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{R}$.*

(i) If $x \in X \backslash \{0\}$ then there exists some $x^* \in X^*$ with $\|x^*\| = 1$ such that $\|x\| = x^*(x)$

(ii) For every $x \in X$

$$\|x\| = \sup\{|x^*(x)| : \|x^*\| = 1\} = \max\{|x^*(x)| : \|x^*\| = 1\}$$

Proof. For (i), we apply the Hahn-Banach Theorem to $Y := \mathbb{R}x$ and $y^*(ax) = a\,\|x\|$ that fulfills $|y^*(ax)| = \|ax\|$ hence $\|y^*\| = 1$.

(ii) is a consequence of $|x^*(x)| \leq \|x\|\ \|x^*\|$. □

CHAPTER **2**

Banach lattices

2.1 Basic definitions and properties

In this section the reader is introduced to some basic definitions and properties concerning vector and Banach lattices. Most of the content here exposed is from [Wickstead, 2007], whereas a great quantity of the following examples were found in [Meyer-Nieberg, 1991] and [Schaefer, 1974].

2.1.1 Introduction to vector lattices

Definition 2.1. An *order* on a non-empty set M is a relation " $\leq$ " such that

(i) $x \leq x \quad \forall x \in M$ (reflexive)

(ii) $x \leq y$ and $y \leq x$ imply that $x = y \quad \forall x, y \in M$ (antisymmetric)

(iii) $x \leq y$ and $y \leq z$ imply that $x \leq z \quad \forall x, y, z \in M$ (transitive)

We use $y \geq x$ as a synonym for $x \leq y$, and $x < y$ for $x \leq y$ but $x \neq y$. Similarly, we write $y > x$ for $x < y$.

Definition 2.2. Let A be a non-empty subset of M, then

(i) $x \in M$ is an *upper bound* for A (resp. *lower bound*) if $y \leq x$ (resp. $x \leq y$) $\forall y \in A$. In this case we say that A is *bounded above* (resp. *bounded below*).

(ii) An upper bound (resp. lower bound) x for A is the *supremum* (resp. *infimum*) of A if for any other upper bound (resp. lower bound) y for A we have $x \leq y$ (resp. $y \leq x$). The supremum of A, when it exists[1], is denoted by $\sup(A)$, $\sup\{a : a \in A\}$, $\bigvee\{a : a \in A\}$, or $\bigvee_{a \in A} a$ (analogous for the infimum).

(iii) If A is bounded above and bounded below, we say that A is *order bounded*.

[1] See Remark 2.10.

(iv) A is an *order interval* if it is of the form $[x, y] := \{m \in M : x \leq m \leq y\}$ for some $x, y \in M$.

Definition 2.3. A *lattice* is a non-empty set $\mathbb{L}$ with an order "$\leq$" such that for every pair of elements $x, y \in \mathbb{L}$, the set $\{x, y\}$ has both a supremum ($x \vee y$) and an infimum ($x \wedge y$) in $\mathbb{L}$.

Example 2.4. $\mathbb{N}$ with the order given by the divisibility ($x \leq y \iff x \mid y$) is a lattice where $x \vee y = mcm(x, y)$ and $x \wedge y = mcd(x, y)$ for every $x, y \in \mathbb{N}$.

Example 2.5. $\mathbb{R}$ with the usual order is a lattice where $x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$ for every $x, y \in \mathbb{R}$.

Example 2.6. Let $\mathbb{M}$ be a non-empty space and let $(\mathbb{L}, \leq)$ be a lattice. The set $\mathbb{L}^{\mathbb{M}}$ of all mappings of the form $f : \mathbb{M} \to \mathbb{L}$ is lattice under its *canonical ordering* ($f \leq g \iff f(t) \leq g(t)\ \forall t \in \mathbb{M}$) where $f \vee g = h$ with $h(t) = f(t) \vee g(t)$ and $f \wedge g = i$ with $i(t) = f(t) \wedge g(t)$ for every $f, g \in \mathbb{L}^{\mathbb{M}}$.

Example 2.7. Let M be the set of all subsets of a certain set X, ordered with inclusion ($A \leq B \iff A \subseteq B$). Then M is a lattice where $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ for every $A, B \in M$.

Definition 2.8. A *sublattice* $\mathbb{A}$ of a lattice $\mathbb{L}$ is simply a vector subspace that is also a lattice. That is, $x, y \in \mathbb{A}$ implies that $x \vee y$, $x \wedge y \in \mathbb{A}$, where these lattice operations are computed in $\mathbb{L}$.

Example 2.9. Let X be an infinite set and N be the collection of all subsets of X such that they are finite or their complement is finite, again ordered by inclusion. After some easy computation it is easy to check that both $A \cup B$ and $A \cap B$ belong to N and hence N is a lattice. In particular, N is a sublattice of the collection of all subsets of X.

Remark 2.10. Let X and N defined as in the previous example and $Y \subset X$ such that $Y \notin N$. Then $B := \{\{x\} : x \in Y\}$ is a subset of N which does not have a supremum in N (note that $\sup(B) = Y$).

Example 2.11. Let X be an infinite set and let $\mathcal{L}$ be the set of all finite subsets of X such that their cardinality is even. Then $\mathcal{L}$ is an ordered set (ordered with inclusion) but fails to be a lattice ($A \wedge B = A \cap B \notin \mathcal{L}$ in general).

Definition 2.12. A *linearly ordered set* (or *linear order*) is a non-empty set $\mathbb{L}$ with an order "$\leq$" with the property that for any $x, y \in \mathbb{L}$, either $x \leq y$ or $y \leq x$.

Remark 2.13. Note that Example 2.5 shows a linearly ordered set.

Definition 2.14. A map $T : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ between two lattices $\mathbb{L}_1$ and $\mathbb{L}_2$ is said to be a *lattice homomorphism* if it preserves the lattice operations:

$$T(x \vee y) = T(x) \vee T(y) \text{ and } T(x \wedge y) = T(x) \wedge T(y) \text{ for every } x, y \in \mathbb{L}_1.$$

If T is also bijective, we say that T is a *lattice isomorphism*.

Definition 2.15. A real vector space E with an order " $\leq$ " is a *vector lattice* (or *Riesz space*) if it is a lattice and also an *ordered vector space* (guarantees compatibility for the order and the vector space):

(i) $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in E$,

(ii) $0 \leq x$ implies $0 \leq tx$ for all $x \in E$ and $0 \leq t \in \mathbb{R}$.

Example 2.16. The most obvious example of a vector lattice is $\mathbb{R}^n$ with all the usual operations and the *usual or standard order* in which

$$(x_1, ..., x_n) \leq (y_1, ..., y_n) \iff x_k \leq y_k \text{ for } k = 1, ..., n.$$

This order makes $\mathbb{R}^n$ into a vector lattice where

$$(x_k) \vee (y_k) = (x_k \vee y_k) = (\max(x_k, y_k)) \text{ and } (x_k) \wedge (y_k) = (x_k \wedge y_k) = (\min(x_k, y_k)).$$

Definition 2.17. A *vector sublattice* of a vector lattice E is simply a vector subspace that is also a sublattice.

Example 2.18. If M is any set, the space of real-valued functions on X, $\mathbb{R}^M$, is a vector lattice under its canonical ordering and pointwise vector operations. Note that for $M = \emptyset$ the usual convention is $\mathbb{R}^M = \{0\}$. Many of the vector lattices occurring in analysis are vector sublattices of $\mathbb{R}^M$.

Example 2.19. Both c_0 and c are vector sublattices of ℓ^∞ (with pointwise order and vector operations).

Example 2.20. If $\{E_\alpha\}_{\alpha \in A}$ is a family of vector lattices, the Cartesian product $\prod_\alpha E_\alpha$ is a vector lattice if the vector and lattice operations are defined coordinatewise. The ordering in $\prod_\alpha E_\alpha$ is called canonical and can be viewed as a generalization of the construction given in Example 2.6. The direct sum $\oplus_\alpha E_\alpha$ of the family $\{E_\alpha\}_{\alpha \in A}$ is understood to be the vector sublattice of $\prod_\alpha E_\alpha$ containing precisely all finitely non-zero elements.

Definition 2.21. The set $E^+ := \{x \in E : x \geq 0\}$ is called the *positive cone* of E and its elements are termed *positive* (rather than non-negative). If $x \in E^+$ is not zero, we will often say that x is *strictly positive*.

Example 2.22. Let X be a non-empty set and let $B(X)$ be the collection of all bounded real valued functions defined on X. As this is a vector subspace of $\mathbb{R}^X$, it is a simple and well-known fact that $B(X)$ is ordered by the positive cone

$$B(X)^+ = \{f \in B(X) : f(t) \geq 0 \text{ for all } t \in X\}.$$

Thus $f \geq g$ holds if and only if $f - g \in B(X)^+$. Obviously,

$$(f \vee g)(t) = \max\{f(t), g(t)\} \text{ and } (f \wedge g)(t) = \min\{f(t), g(t)\}$$

for every $t \in X$ and $f, g \in B(X)$. This shows that $B(X)$ is a vector lattice and, in particular, a vector sublattice of $\mathbb{R}^X$.

Observation 2.23. In Example 2.42 we will see that when K is compact the subspace $C(K)$ is a Banach lattice with the norm $\|\cdot\|_\infty$ and in particular a vector sublattice of $B(K)$.

Example 2.24. Let X be an infinite set and let $\mathcal{L} = \mathcal{P}(X)$. Then

$$E = span\{\chi_A : A \in \mathcal{L}\} = \{f\colon X \to \mathbb{R} \text{ where the image of } f \text{ is finite}\} =$$

$$\{\lambda_1 \cdot 1_{A_1} + ... + \lambda_n \cdot 1_{A_n} : \lambda_i \in \mathbb{R} \text{ arbitrary and } A_1, ..., A_n \text{ is a partition of } X\}$$

is a vector sublattice of $B(X)$.

Example 2.25. Now let X be an infinite set and let $\mathcal{L}$ be all finite subsets of X with even cardinality. Then $E = span\{\chi_A : A \in \mathcal{L}\}$ is an ordered vector space (subset of $B(X)$) but fails to be a vector lattice as it is not closed for suprema and infima.

Definition 2.26. For any $x \in E$, we define the *positive part* of x as $x^+ := x \vee 0$, the *negative part* of x as $x^- := (-x) \vee 0$, and the *modulus* of x as $|x| := x \vee (-x)$.

Observation 2.27. Recalling Example 2.16, it is obvious that the positive part, the negative part and the modulus are given by $(x_k)^+ = (x_k^+)$, $(x_k)^- = (x_k^-)$ and $|(x_k)| = (|x_k|)$, respectively.

Definition 2.28. We say that $x, y \in E$ are *disjoint*, $x \perp y$, if $|x| \wedge |y| = 0$. If $A \subset E$, then $A^d = \{y \in E : y \perp a \;\forall a \in A\}$.

Lemma 2.29. *For all x, y in a vector lattice, we have $x + y = (x \vee y) + (x \wedge y)$.*

Proof. For any given x and y in a vector lattice, let $w = (-y) \vee (-x)$ and $v = x \vee y$. Obviously we have $w = -(x \wedge y)$. In view of $w \geq -y$ and $w \geq -x$ it follows that $w + x + y \geq x$ and $w + x + y \geq y$. Consequently, $w + x + y \geq x \vee y = v$, that is, $x + y \geq x \vee y + x \wedge y$ for every x, y.

Multiplying each side by -1, we get $-x - y \leq -(x \vee y) - (x \wedge y) = (-x) \vee (-y) + (-x) \wedge (-y)$ for every $-x, -y$. Equivalently, $x + y \leq x \vee y + x \wedge y$ for every x, y.

□

Lemma 2.30. *Let x_1 and x_2 be two positive disjoint elements in a vector lattice. Then, if c_1 and c_2 are two positive real numbers, c_1x_1 and c_2x_2 are disjoint.*

Proof. First, note that any positive element is equal to its modulus. Now we know that

$$0 \leq c_1x_1 \wedge c_2x_2 \leq [(c_1 \vee c_2)x_1] \wedge [(c_1 \vee c_2)x_2] = (c_1 \vee c_2)(x_1 \wedge x_2) = (c_1 \vee c_2) \cdot 0 = 0$$

hence $c_1x_1 \wedge c_2x_2 = 0$ as we wanted to prove. □

Lemma 2.31. *Let $x_1, x_2, ..., x_n$ be positive pairwise disjoint elements in a vector lattice. Then, if $c_1, c_2, ..., c_n$ are positive real numbers, we have $\sum_{i=1}^n c_ix_i = \bigvee_{i=1}^n c_ix_i$.*

Proof. We will prove it using induction on the quantity n.

For $n = 2$, we have

$$c_1x_1 + c_2x_2 = (c_1x_1 \vee c_2x_2) + (c_1x_1 \wedge c_2x_2) = c_1x_1 \vee c_2x_2$$

by Lemmas 2.30 and 2.29.

Now let us suppose that the result holds for some $n \geq 2$. Then we have

$$\sum_{i=1}^{n+1} c_ix_i = \sum_{i=1}^{n} c_ix_i + c_{n+1}x_{n+1} = \bigvee_{i=1}^{n} c_ix_i + c_{n+1}x_{n+1} = \bigvee_{i=1}^{n} c_ix_i \vee c_{n+1}x_{n+1} = \bigvee_{i=1}^{n+1} c_ix_i.$$

Where we used Lemma 2.29 and the fact that $\bigvee_{i=1}^{n} c_ix_i$ and $c_{n+1}x_{n+1}$ are positive and disjoint by Lemma 2.30, as

$$\bigvee_{i=1}^{n} c_ix_i \wedge c_{n+1}x_{n+1} = (c_1x_1 \wedge c_{n+1}x_{n+1}) \vee ... \vee (c_nx_n \wedge c_{n+1}x_{n+1}) = 0 \vee ... \vee 0 = 0.$$

□

Proposition 2.32. *Let $x_1, x_2, ..., x_n$ be positive pairwise disjoint elements in a vector lattice. Then, if $c_1, c_2, ..., c_n, d_1, d_2, ..., d_n$ are positive real numbers and $x = \sum_{i=1}^{n} c_ix_i$ and $y = \sum_{i=1}^{n} d_ix_i$, we have*

$$x \vee y = \sum_{i=1}^{n}(c_i \vee d_i)x_i \quad \textit{and} \quad x \wedge y = \sum_{i=1}^{n}(c_i \wedge d_i)x_i.$$

Proof. By Lemma 2.31, it is

$$x \vee y = \left(\bigvee_{i=1}^{n} c_ix_i\right) \vee \left(\bigvee_{i=1}^{n} d_ix_i\right) = (c_1x_1 \vee d_1x_1) \vee ... \vee (c_nx_n \vee d_nx_n) =$$
$$(c_1 \vee d_1)x_1 \vee ... \vee (c_n \vee d_n)x_n = \sum_{i=1}^{n}(c_i \vee d_i)x_i.$$

Hence, by Lemma 2.29, we also have

$$x \wedge y = x + y - x \vee y = \sum_{i=1}^{n}(c_i + d_i - c_i \vee d_i)x_i = \sum_{i=1}^{n}(c_i \wedge d_i)x_i.$$

□

Lemma 2.33. *For all x, y, z in a vector lattice, we have $(x \vee y) + z = (x + z) \vee (y + z)$.*

Proof. First, $(x \vee y) + z \geq x + z$ and $(x \vee y) + z \geq y + z$, so it is obvious that $(x \vee y) + z \geq (x + z) \vee (y + z)$.

Now let us assume that $(x \vee y) + z > (x + z) \vee (y + z)$. Then there must be some element w' such that $(x \vee y) + z \geq w' = w + z \geq (x + z) \vee (y + z)$. In particular, $(x \vee y) + z \geq w + z \geq x + z$ and $(x \vee y) + z \geq w + z \geq y + z$ so this means there exists some w such that $x \vee y \geq w \geq x$ and $x \vee y \geq w \geq y$ which is absurd by definition of supremum. □

Proposition 2.34. *Let $x_1, x_2, ..., x_n$ be positive pairwise disjoint elements in a vector lattice. Then, for some $c_1, c_2, ..., c_n, d_1, d_2, ..., d_n \in \mathbb{R}$, if $x = \sum_{i=1}^n c_i x_i$ and $y = \sum_{i=1}^n d_i x_i$, we have*

$$x \vee y = \sum_{i=1}^n (c_i \vee d_i) x_i \quad \text{and} \quad x \wedge y = \sum_{i=1}^n (c_i \wedge d_i) x_i.$$

Proof. Let us take some big enough $\alpha \in \mathbb{R}$ such that $c_i + \alpha, d_i + \alpha > 0$ for every $i \in \{1, ..., n\}$. Then, as

$$x = \sum_{i=1}^n c_i x_i = \sum_{i=1}^n (c_i + \alpha) x_i - \sum_{i=1}^n \alpha x_i \quad \text{and} \quad y = \sum_{i=1}^n d_i x_i = \sum_{i=1}^n (d_i + \alpha) x_i - \sum_{i=1}^n \alpha x_i,$$

by Proposition 2.32 and Lemma 2.33 we have

$$x \vee y = \left(\sum_{i=1}^n (c_i + \alpha) x_i \right) \bigvee \left(\sum_{i=1}^n (d_i + \alpha) x_i \right) - \sum_{i=1}^n \alpha x_i =$$
$$\left(\sum_{i=1}^n [(c_i + \alpha) \vee (d_i + \alpha)] x_i \right) - \sum_{i=1}^n \alpha x_i = \left(\sum_{i=1}^n (c_i \vee d_i) + \alpha x_i \right) - \sum_{i=1}^n \alpha x_i = \sum_{i=1}^n (c_i \vee d_i) x_i.$$

Hence, by Lemma 2.29, we also have

$$x \wedge y = x + y - x \vee y = \sum_{i=1}^n (c_i + d_i - c_i \vee d_i) x_i = \sum_{i=1}^n (c_i \wedge d_i) x_i.$$

□

Corollary 2.35. *Let $x_1, x_2, ..., x_n$ be positive pairwise disjoint elements in a vector lattice. Then, for some $c_1, c_2, ..., c_n \in \mathbb{R}$, if $x = \sum_{i=1}^n c_i x_i$, we have*

$$|x| = \sum_{i=1}^n |c_i| \cdot x_i$$

Proof. It is a consequence of Proposition 2.34 where $y = -x$. □

Definition 2.36. A map $T \colon E_1 \longrightarrow E_2$ between two vector lattices E_1 and E_2 is said to be a *vector lattice homomorphism* if it is linear and preserves the lattice operations:

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \text{ for every } x, y \in E_1, \alpha, \beta \in \mathbb{R},$$

$$T(x \vee y) = T(x) \vee T(y) \text{ and } T(x \wedge y) = T(x) \wedge T(y) \text{ for every } x, y \in E_1.$$

If T is also bijective, we will say that T is a *vector lattice isomorphism*.

2.1.2 Introduction to Banach lattices

Definition 2.37. A *Banach lattice* is a vector lattice together with a norm that is also a Banach space in which $|x| \leq |y| \implies \|x\| \leq \|y\|$ (i.e. $\|\cdot\|$ is a *lattice norm*).

Definition 2.38. A *Banach sublattice* of a Banach lattice is simply a vector subspace which is also a sublattice and closed under the norm of the Banach lattice. In particular, a Banach sublattice is clearly a Banach lattice.

Lemma 2.39. *Let E be a Banach lattice. For any $x \in E$ it is $\|x\| = \|\,|x|\,\|$.*

Proof. It is a consequence of the fact that $|x| = |\,|x|\,|$ and that in a Banach lattice $(|x| \leq |y| \implies \|x\| \leq \|y\|)$ holds. □

Definition 2.40. The Banach space dual E^* of a Banach lattice E is a Banach lattice with respect to the order given by

$$x^* \geq 0 \text{ if and only if } x^*(x) \geq 0 \text{ for every } x \geq 0.$$

All elements that satisfy the inequality above are called *positive*.

The infimum $\wedge$ and the supremum $\vee$ of two elements $x^*, y^* \in E^*$ are given by

$$(x^* \wedge y^*)(x) = \inf\{x^*(x_1) + y^*(x_2) : x = x_1 + x_2,\ x_1, x_2 > 0\} \text{ for every } x > 0$$

$$(x^* \vee y^*)(x) = \sup\{x^*(x_1) + y^*(x_2) : x = x_1 + x_2,\ x_1, x_2 > 0\} \text{ for every } x > 0$$

while the modulus of an element $x^* \in E^*$ is given by

$$|x^*|(x) = \sup\{x^*(y) : |y| \leq x\} \text{ for every } x > 0.$$

Example 2.41. ℓ^p, c_0 and c are Banach lattices with their usual norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$ and under pointwise order.

Example 2.42. The space of continuous real functions on K, $C(K)$, where K is compact, is a Banach lattice with the norm $\|\cdot\|_\infty$ and the pointwise order.

Example 2.43. $L_p(\mu)$ is a Banach lattice with the norm $\|\cdot\|_p$ and under pointwise almost everywhere order for $p \geq 1$.

Definition 2.44. A map $T\colon X \longrightarrow Y$ between two Banach lattices X and Y is said to be a *Banach lattice homomorphism* if it is a bounded linear operator and preserves the lattice operations.

If T is also bijective and T^{-1} is a Banach lattice homomorphism, we will say T is a *Banach lattice isomorphism*.

If, moreover, T preserves the norm ($\|T(x)\| = \|x\|$ for every $x \in X$), we will say that T is a *Banach lattice isometry*.

2.2 Elementary inequalities and the Archimedean property

Definition 2.45. An *elementary inequality* in a vector lattice is an inequality or an equality that involves linear and lattice operations and a finite number of elements of the vector lattice, which can be true or false.

Example 2.46. The equality $x + (y \vee z) = (x + y) \vee (x + z)$ is an elementary inequality.

Definition 2.47. A vector lattice E has the *Archimedean property* (or is *Archimedean*) if

$$nx \leq y \text{ for all } n \in \mathbb{N} \text{ and } y \in E^+ \text{ implies } x \leq 0.$$

Example 2.48. Clearly, $\mathbb{R}^n$ with the usual order is Archimedean.

Example 2.49. $\mathbb{R}^2$ with the *lexicographic* order given by $(x_1, x_2) \leq (y_1, y_2) \iff x_1 < y_1$ or $x_1 = y_1$ and $x_2 \leq y_2$ fails to have the Archimedean property as $n(0, 1) \leq (1, 0)$ for all $n \in \mathbb{N}$ but $(0, 1) \not\leq (0, 0)$.

Example 2.50. Real-valued function spaces defined on non-empty sets X, $E = \mathbb{R}^X$, ordered with their canonical order and under pointwise vector operations are important examples of Archimedean vector lattices. E is Archimedean as if $nf \leq g$ for all $n \in \mathbb{N}$ then $nf(x) \leq g(x)$ for all $n \in \mathbb{N}$ and for all $x \in X$ and from the fact that $\mathbb{R}$ is Archimedean follows that $f(x) \leq 0$ for all $x \in X$ and hence that $f \leq 0$ (where this 0 is the zero function on X).

E will have many vector subspaces which are also vector lattices under the same order, and these will obviously be Archimedean too. For example, we could take the bounded functions and, if X has a topology, the continuous functions or continuous bounded functions.

Theorem 2.51 ([Meyer-Nieberg, 1991])**.** *An elementary inequality is true in every Archimedean vector lattice if and only if it is true in the reals.*

The idea of the proof relies on the result being true for uniformly complete vector lattices (page 66 of [Meyer-Nieberg, 1991]) and the fact that all Archimedean vector lattices may be embedded as sublattices of a certain uniformly complete vector lattice ([Schaefer, 1974, Proposition 1.10]), their Dedekind completion, from which the general result follows.

Proposition 2.52. *Every normed vector lattice E is Archimedean. In particular, every Banach lattice is Archimedean.*

Proof. Consider $x, y \in E$ such that $nx \leq y$ for all $n \in \mathbb{N}$. Hence $x^+ \leq n^{-1}y^+$ for every $n \in \mathbb{N}$. If E is a normed vector lattice, then $\|x^+\| \leq n^{-1} \|y^+\|$. Hence $x^+ = 0$ and $x \leq 0$. □

Corollary 2.53. *If an elementary inequality is true in the reals, then it is true in every Banach lattice.*

Proof. This is an immediate consequence of Theorem 2.51 and Proposition 2.52 □

2.3 Ideals and bands

In this section we will introduce ideals and bands, two very important concepts on vector lattices. Most of the definitions shown below can be found in [Wickstead, 2007] and some proofs for the following properties related to them that exceeded the interest of this Chapter are in Section 1.2 of [Meyer-Nieberg, 1991]. In the end of this Section we will get to a specific corollary from a theorem, Freudenthal's Spectral theorem, that will be essential for one of our main results from Chapter 4.

Definition 2.54. An *ideal* J in a vector lattice E is a vector subspace such that $y \in J$, $x \in E$ and $|x| \leq |y|$ together imply that $x \in J$.

Example 2.55. c_0 is an ideal in l_∞ because of the Squeeze Theorem but c is not as $|((-1)^n)| \leq (1) \in c$ but $((-1)^n) \notin c$.

Definition 2.56. Let A be a non-empty subset of a vector lattice E. The *ideal generated by* A is the smallest ideal in E containing A.

Definition 2.57. A *principal ideal* in a vector lattice E is an ideal J which is generated by a single element i.e., there exists $e \in E_+$ such that

$$J = \{x : |x| \leq ne \text{ for some } n \in \mathbb{N}\} = \bigcup_{n \in \mathbb{N}} [-ne, ne],$$

which is often denoted by E_e. If $e \in E$ is such that $E_e = E$, e is termed an *order unit* for E.

Definition 2.58. If e is an order unit for E then the expression

$$\|x\|_e = inf\{\lambda \geq 0 : |x| \leq \lambda e\}$$

is a norm on the Archimedean vector lattice E and is referred to as the *order unit norm generated by* e. Remember that on Proposition 2.52 we saw that any Banach lattice is Archimidean. If E was not Archimedean, this would only be a semi-norm.

Example 2.59. The supremum norm on $C(K)$ is precisely the order unit norm generated by 1_K.

Proposition 2.60. *Assume that E is a Banach lattice and let $e \in E_+$ be an order unit for E. In this case the norms $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent.*

Proof. See [Meyer-Nieberg, 1991, 1.2.14]. □

Definition 2.61. A *band* in a vector lattice E is an ideal J with the property that if $A \subseteq J$ and x is the supremum of A in E then $x \in J$.

Example 2.62. If I is any subset of $\mathbb{N}$ then $\{(x_n) \in l_\infty : x_n = 0 \ \forall n \in I\}$ is a band in l_∞, but c_0 is not a band as the unit ball in c_0 has supremum (1) in l_∞ which does not lie in c_0.

Definition 2.63. A *principal band* is one which is generated by a single element i.e., a principal ideal that is a band.

Definition 2.64. A band B in a vector lattice E is a *projection band* if $E = B \oplus B^d$. In this case the linear projection $P_B\colon E \to B$ that takes $x + y$ to x for any $x \in B, y \in B^d$ is the *band projection onto B*.

Proposition 2.65. *Let B_{z_1} and B_{z_2} be two principal projection bands in a vector lattice E generated by disjoint elements z_1 and z_2, and let P_{z_1} and P_{z_2} be their respective band projections. Then if the principal band in E generated by $z_1 + z_2$ is a projection band, its band projection is $P_{z_1} + P_{z_2}$.*

Proof. First, we fix $x \in E$ and express $x = P_{z_1}(x)+P_{z_2}(x)+y$ where $y = x-P_{z_1}(x)-P_{z_2}(x)$. We just need to prove that $y \in B^d_{z_1+z_2}$ and therefore the result follows.

For any $z_1' \in B_{z_1}$ and $z_2' \in B_{z_2}$ it is

$$0 \leq |z_1'| \wedge |P_{z_2}(x)| \leq n_1' z_1 \wedge n_2 z_2$$
$$0 \leq |z_2'| \wedge |P_{z_1}(x)| \leq n_2' z_2 \wedge n_1 z_1$$

for some positive numbers $n_1, n_2, n_1', n_2' \in \mathbb{N}$. Now by Lemma 2.30, $n_1' z_1 \wedge n_2 z_2 = 0$ and $n_2' z_2 \wedge n_1 z_1 = 0$ so consequently $z_1' \wedge P_{z_2}(x) = 0$ and $z_2' \wedge P_{z_1}(x) = 0$. Thus, $P_{z_2}(x) \in B^d_{z_1}$ and $P_{z_1}(x) \in B^d_{z_2}$.

By definition of band projection we have

$$\begin{cases} x - P_{z_1}(x) = y + P_{z_2}(x) \in B^d_{z_1} \\ P_{z_2}(x) \in B^d_{z_1} \end{cases} \Rightarrow y \in B^d_{z_1}$$

$$\begin{cases} x - P_{z_2}(x) = y + P_{z_1}(x) \in B^d_{z_2} \\ P_{z_1}(x) \in B^d_{z_2} \end{cases} \Rightarrow y \in B^d_{z_2}.$$

Hence, as $B^d_{z_1} \cap B^d_{z_2} = B^d_{z_1 \vee z_2}$ ($|z| \wedge (z_1 \vee z_2) = (|z| \wedge z_1) \vee |z| \wedge z_2$), it is $y \in B^d_{z_1 \vee z_2} = B^d_{z_1+z_2}$, where we used Lemma 2.29 and the fact that z_1 and z_2 are positive and disjoint. □

Definition 2.66. A vector lattice E is called *σ-Dedekind complete* if every order bounded increasing sequence has a supremum in E.

Definition 2.67. If every principal band in a vector lattice E is a projection band, then E is said to have the *principal projection property*.

Proposition 2.68. *Every dual Banach lattice E^* is σ-Dedekind complete. In particular, every bidual Banach lattice E^{**} is σ-Dedekind complete.*

Proof. Let us consider an order bounded above sequence in E^*

$$x_1^* \leq x_2^* \leq ... \leq x_n^* \leq ... \leq x_0^*.$$

Then for every $x \in E$ we have

$$x_1^*(x) \leq x_2^*(x) \leq ... \leq x_n^*(x) \leq ... \leq x_0^*(x),$$

which is an increasing and bounded above sequence in $\mathbb{R}$ and hence it converges to its supremum. Thus, we are able to define $x^*(x) := \sup x_n^*(x) = \lim x_n^*(x)$ which is a valid supremum for our original sequence. □

Proposition 2.69. *Every σ-Dedekind complete vector lattice has the principal projection property.*

Proof. See [Meyer-Nieberg, 1991, 1.2.11]. $\square$

Corollary 2.70. *Every bidual Banach lattice E^{**} has the principal projection property.*

Proof. It is a direct consequence of Propositions 2.68 and 2.69. $\square$

Definition 2.71. For every $e \in E_+$ let $Q(e) = \{v \in E_+ : v \wedge (e - v) = 0\}$ be the collection of all *quasi units with respect to* e.

Proposition 2.72. *For a vector lattice E, we fix $0 \neq e \in E_+$.*

(i) $u \vee v \in Q(e)$ *for all* $u, v \in Q(e)$.

(ii) *For all $u, v \in Q(e)$ let $\delta(u, v) = 1$ if $u \wedge v \neq 0$ and $\delta(u, v) = 0$ if $u \wedge v = 0$.*
If $v_1, ..., v_r \in Q(e)$, then there exist finitely many disjoint $z_1, ..., z_m \in Q(e)$ such that

$$v_j = \sum_{i=1}^{m} \delta(v_j, z_i) z_i$$

for all $j = 1, ..., r$.

Proof. See [Meyer-Nieberg, 1991, 1.2.17]. $\square$

2.3.1 Freudenthal's Spectral theorem

Theorem 2.73 (Freudenthal's Spectral theorem)**.** *Assume that the vector lattice E has the principal projection property and $0 \neq e \in E_+$. For every $x \in E_e$ and $\epsilon > 0$ there exist finitely many $v_{-r}, ..., v_{-1}, v_0, v_1, ..., v_r \in Q(e)$ and $m \in \mathbb{N}$ such that*

$$\left\| \frac{m}{r} \sum_{i=1-r}^{r} i(v_i - v_{i-1}) - x \right\|_e \leq \epsilon.$$

Proof. See [Meyer-Nieberg, 1991, 1.2.18]. $\square$

By all the aforementioned definitions and consequent properties, we are able to derive a better version of the theorem that will be useful later when proving our main result.

Note that we can use Proposition 2.72 (ii) to transform the set $v_{-r}, ..., v_{-1}, v_0, v_1, ..., v_r$ from Theorem 2.73 into the pairwise disjoint set $z_1, \ldots, z_m$, such that every v_i can be expressed as a sum of elements of it. Then by Proposition 2.72 (i) and Lemma 2.29 we know that $z' = z_1 + ... + z_m = z_1 \vee ... \vee z_m \in Q(e)$ and hence the element $z_{m+1} = e - z'$ is also in $Q(e)$.

Corollary 2.74. *Let E be a Banach lattice. Suppose we have $\epsilon > 0$ and $0 \neq e \in E_+^{**}$. Then, for every $x \in E_e^{**}$ there exist pairwise disjoint $z_1, \ldots, z_{m+1} \in E_+^{**}$ with $e = z_1 + \cdots + z_{m+1}$ and*

$$d(x, span\{z_1, \ldots, z_{m+1}\}) < \epsilon.$$

2.4 Hahn-Banach theorem for positive operators

Theorem 2.75 (Extension for Positive Operators). *Let F be a Banach lattice and U be a sublattice of F. Every positive linear operator $T\colon U \to \mathbb{R}$ such that $T(x) \leq \|x\|$ for every $x \in U$ extends to a positive linear operator $S\colon F \to \mathbb{R}$ satisfying $\|S\| \leq 1$.*

Proof. See [Meyer-Nieberg, 1991, 1.5.7]. □

Corollary 2.76. *Let F be a Banach lattice and let K be the positive part of the closed unit ball of F^*, B_{F^*}.*

(i) If $x \in F_+$ then there exists some positive operator $x^* \in F^*$ with $\|x^*\| \leq 1$, that is, some positive operator $x^* \in K$, such that $\|x\| = x^*(x)$

(ii) For every $x \in F$

$$\|x\| = \|\,|x|\,\| = \sup\{x^*(|x|) : x^* \in K\} = \max\{x^*(|x|) : x^* \in K\}$$

Proof. For (i), we apply the Extension for Positive Operators Theorem to $U := Ix$ where $I = [0,1] \subset \mathbb{R}$ and $T(ax) = a\,\|x\| \leq \|x\|$.

(ii) is a consequence of $|x^*(x)| \leq \|x\|\,\|x^*\|$ and Lemma 2.39. □

CHAPTER **3**

Universal injective Banach space for the class of all separable Banach spaces

3.1 Banach-Alaoglu theorem

In this chapter some results that are needed for its main theorem, the Banach-Mazur theorem, will be introduced. All of these were found in [Albiac and Kalton, 2016].

Lemma 3.1. *Let X be a normed vector space. The closed unit ball of X^*, B_{X^*}, endowed with the relative weak* topology is a Hausdorff space.*

Proof. The dual space X^* endowed with the weak* topology is Hausdorff because for any $x^*, y^* \in X^*$ such that $x^* \neq y^*$ there must exist some $z \in X$ such that $x^*(z) \neq y^*(z)$ so then we can assume without loss of generality that there is some ζ with $x^*(z) < \zeta < y^*(z)$. Hence, the open sets defined as $\{w^* \in X^* : w^*(z) < \zeta\}$ and $\{w^* \in X^* : w^*(z) > \zeta\}$ are disjoint neighborhoods of x^* and y^*, respectively. As a consequence, by Proposition 1.11, B_{X^*} is also Hausdorff. □

Lemma 3.2. *Let X be a separable space. The closed unit ball of X, B_X, is also a separable space.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be dense in X. We state that $(x_n)_{n=1}^{\infty} \cap \hat{B}$ is dense in B_X, where $\hat{B}$ is the open unit ball of X. Let V be an arbitrary open subset of B_X, then $V = U \cap B_X$ where U is an open subset of X. Then $U \cap \hat{B} \subset V$ is open in X so there exists some $x_n \in U \cap \hat{B} \subset V$ as we wanted to see. □

Lemma 3.3. *Let $(X, \|\cdot\|)$ a normed linear space over $\mathbb{R}$ and $y^* \colon B_X \to \mathbb{R}$ a bounded mapping such that:*

(i) $y^*(\lambda x) = \lambda y^*(x)$ *for every* $x \in B_X$ *and real* λ *such that* $\lambda x \in B_X$.

(ii) $y^*(x_1 + x_2) = y^*(x_1) + y^*(x_2)$ *for every* $x_1, x_2 \in B_X$ *such that* $x_1 + x_2 \in B_X$

Then there exists some $z^*\colon X \to \mathbb{R}$ bounded linear operator ($z^* \in X^*$) such that $z^*|B_X = y^*$.

Proof. We state that the mapping $z^*\colon X \to \mathbb{R}$ defined as $z^*(x) = \|x\| \cdot y^*(\frac{x}{\|x\|})$ when $x \neq 0$ and $z^*(0) = 0$ accomplishes the conditions aforementioned. First, as $\left\|\frac{x}{\|x\|}\right\| = \frac{1}{\|x\|} \cdot \|x\| = 1$, then $\frac{x}{\|x\|} \in B_X$ and it is well defined. If $x \in B_X$ then $\|x\| = 1$ and $z^*(x) = y^*(x)$ so $z^*|B_X = y^*$. Moreover, z^* is bounded because $\|z^*\| = sup_{x\in B_X}\{z^*(x)\} = sup_{x\in B_X}\{y^*(x)\} = \|y^*\|$ and y^* is bounded. Lastly, we check that z^* is a linear operator:

- For any $x \in X$ and real λ,

$$z^*(\lambda x) = \|\lambda x\| \cdot y^*(\frac{\lambda x}{\|\lambda x\|}) = |\lambda| \cdot \frac{\lambda}{|\lambda|} \cdot \|x\| \cdot y^*(\frac{x}{\|x\|}) = \lambda \cdot z^*(x)$$

  where we used that $\frac{\lambda x}{\|\lambda x\|} = \frac{\lambda}{|\lambda|} \cdot \frac{x}{\|x\|} \in B_X$

- For any $x_1, x_2 \in X$,

$$z^*(x_1 + x_2) = \|x_1 + x_2\| \cdot y^*(\frac{x_1 + x_2}{\|x_1 + x_2\|}) = \|x_1 + x_2\| \cdot (y^*(\frac{x_1}{\|x_1 + x_2\|}) + y^*(\frac{x_2}{\|x_1 + x_2\|}))$$
$$= \|x_1\| \cdot y^*(\frac{x_1}{\|x_1\|}) + \|x_2\| \cdot y^*(\frac{x_2}{\|x_2\|})) = z^*(x_1) + z^*(x_2)$$

  where we used that $\frac{x_1}{\|x_1+x_2\|}, \frac{x_2}{\|x_1+x_2\|} \in B_X$ and that $\frac{x_1}{\|x_1+x_2\|} = \frac{\|x_1\|}{\|x_1+x_2\|} \cdot \frac{x_1}{\|x_1\|}$, $\frac{x_2}{\|x_1+x_2\|} = \frac{\|x_2\|}{\|x_1+x_2\|} \cdot \frac{x_2}{\|x_2\|}$.

□

Theorem 3.4 (Banach-Alaoglu theorem). *If X is a Banach space then the closed unit ball of X^*, B_{X^*}, is compact for the relative weak* topology.*

Proof. We first consider the function

$$\begin{aligned} \delta_{B_X}\colon X^* &\to \mathbb{R}^{B_X} \\ x^* &\mapsto (x^*(x))_{x\in B_X} \end{aligned}$$

This is continuous for the weak* and the product topologies because every component function is continuous by definition of weak* topology (see 1.41).

Now as $|x^*(x)| \leq \|x^*\| \cdot \|x\| = 1$, by taking the restriction $\delta_{B_X}|_{B_{X^*}}$ we obtain the continuous mapping

$$\begin{aligned} \delta_{B_X}|_{B_{X^*}}\colon B_{X^*} \subset X^* &\to [-1,1]^{B_X} \subset \mathbb{R}^{B_X} \\ x^* &\mapsto (x^*(x))_{x\in B_X} \end{aligned}$$

Let $\hat{O}$ be an open set for the relative weak* topology on B_{X^*}. Then $\hat{O}$ must be a union of basic open sets, so $\hat{O} = \cup_{n=1}^{\infty} B_n \cap B_{X^*}$ where B_n is basic on X^*. Its image by $\delta_{B_X}|_{B_{X^*}}$ is the set $\{(x^*(x))_{x\in B_X} : x^* \in \hat{O}\} \subset [-1,1]^{B_X}$ but, more precisely, it is

$$\{(x^*(x))_{x\in B_X} : x^* \in \cup_{n=1}^{\infty} B_n\} \cap [-1,1]^{B_X} = \cup_{n=1}^{\infty}\{(x^*(x))_{x\in B_X} : x^* \in B_n\} \cap [-1,1]^{B_X}.$$

As every term $\{(x^*(x))_{x\in B_X} : x^* \in B_n\} \cap [-1,1]^{B_X}$ is the intersection of an open set in the product topology (definition 1.41) and $[-1,1]^{B_X}$, it is an open set in the relative product topology and its union must be open. Thus, $\delta_{B_X}|_{B_{X^*}}$ is open.

Lastly, the function $\delta_{B_X}|_{B_{X^*}}$ must be injective because if $x^* \neq y^*$ then there must be some φ in X such that $x^*(\varphi) \neq y^*(\varphi)$. If $x^*(x) = y^*(x)$ for every $x \in B_X$, we have

$$x^*(\varphi) = \|\varphi\| \cdot x^*(\frac{\varphi}{\|\varphi\|}) = \|\varphi\| \cdot y^*(\frac{\varphi}{\|\varphi\|}) = y^*(\varphi)$$

which is an absurd. Note that here we used the fact that $\frac{\varphi}{\|\varphi\|} \in B_X$.

Now by Proposition 1.18, $\delta_{B_X}|_{B_{X^*}}$ is an homeomorphism into its image and so the unit ball of X^* is homeorphic to a subset K of $[-1,1]^{B_X}$, which is compact by Tychonoff's Theorem (1.16).

The fact that K is closed in $[-1,1]^{B_X}$ will finish our reasoning, as this means that B_{X^*} is compact on the weak* topology by Proposition 1.6 and the fact that compactness is a topological property (see Proposition 1.9).

But it is easy to see that if $y^* \in [-1,1]^{B_X} \backslash \delta_{B_X}(B_{X^*})$ then it must be a (bounded) mapping $y^* \colon B_X \to [-1,1]$ with $y^* \neq z^*|_{B_X}$ for any $z^* \in X^*$. By Lemma 3.3, this is equivalent to split this condition into two feasible cases:

- $y^*(\lambda x) \neq \lambda y^*(x)$ for some $x \in B_X$ and real λ such that $\lambda x \in B_X$. By defining the set of mappings $\{z^* \colon B_X \to [-1,1] : z^*(\lambda x) \in (y^*(\lambda x) - \epsilon, y^*(\lambda x) + \epsilon), z^*(x) \in (y^*(x) - \epsilon, y^*(x) + \epsilon)\}$ for a certain $\epsilon > 0$ such that $z^*(\lambda x) \neq \lambda z^*(x)$, we obtain a basic open set in the product topology of $[-1,1]^{B_X}$ that contains y^* and is contained in $[-1,1]^{B_X} \backslash \delta_{B_X}(B_{X^*})$, as we wanted to prove.

- $y^*(x_1 + x_2) \neq y^*(x_1) + y^*(x_2)$ for some $x_1, x_2 \in B_X$ such that $x_1 + x_2 \in B_X$. Again, the set of mappings $\{z^* \colon B_X \to [-1,1] : z^*(x_1 + x_2) \in (y^*(x_1 + x_2) - \epsilon, y^*(x_1 + x_2) + \epsilon), z^*(x_1) \in (y^*(x_1) - \epsilon, y^*(x_1) + \epsilon), z^*(x_2) \in (y^*(x_2) - \epsilon, y^*(x_2) + \epsilon)\}$ for a certain value $\epsilon > 0$ is an open neighborhood of y^*.

Hence, K is closed, and this completes the proof. □

Theorem 3.5. *If X is a separable Banach space then the closed unit ball of X^*, B_{X^*}, is metrizable for the relative weak* topology.*

Proof. Let us take $D := (x_i)_{i=1}^{\infty}$ dense in the unit ball B_X of X (see Lemma 3.2). Note that the proof relies on Corollary 1.22.

We define the topology ρ induced on X^* by convergence on each x_i. Precisely, a base of neighborhoods for ρ at a point $x_0^* \in X^*$ is given by sets of the form

$$V_\epsilon(x_0^*; x_1, ..., x_N) = \{x^* \in X^* : |x^*(x_i) - x_0^*(x_i)| < \epsilon, i = 1, ..., N\},$$

where $\epsilon > 0$ and $N \in \mathbb{N}$. This topology is clearly weaker than the weak* topology, as its basic elements belong to the weak* topology basis (see Definition 1.41), and is Hausdorff on B_{X^*} by the same reasoning as in Lemma 3.1, so it coincides with the weak* topology on the weak* compact set B_{X^*} (Banach-Alaoglu, Theorem 3.4).

If we now define the mapping

$$\begin{aligned} \delta_D \colon X^* &\to \mathbb{R}^D \\ x^* &\mapsto (x^*(x_i))_{i=1}^{\infty} \end{aligned}$$

we can state that it will be continuous by the same argument as in the previous proof: every component function is continuous by definition of ρ topology.

Now we see that the function δ_D must be injective because if $x^* \neq y^*$ then there must be some φ in X such that $x^*(\varphi) \neq y^*(\varphi)$. If $x^*(x_i) = y^*(x_i)$ for every x_i in D, then we have

$$\begin{aligned} &|x^*(\frac{\varphi}{\|\varphi\|}) - y^*(\frac{\varphi}{\|\varphi\|})| \leq |x^*(\frac{\varphi}{\|\varphi\|}) - x^*(x_i)| + |y^*(x_i) - y^*(\frac{\varphi}{\|\varphi\|})| \\ &= |x^*(\frac{\varphi}{\|\varphi\|} - x_i)| + |y^*(x_i - \frac{\varphi}{\|\varphi\|})| \end{aligned}$$

As the set D is dense in B_X, we can always find some x_i such that $|x_i - \frac{\varphi}{\|\varphi\|}| < \epsilon$ for any given $\epsilon > 0$. Hence, making it tend to zero we would have $x^*(\frac{\varphi}{\|\varphi\|}) = y^*(\frac{\varphi}{\|\varphi\|})$ or equivalently $x^*(\varphi) = y^*(\varphi)$ which is an absurd.

By Theorem 1.21, $\delta_D|_{B_{X^*}}$ is a topological embedding and (B_{X^*}, ρ) is homeomorphic to some subset of $\mathbb{R}^D$. This gives that ρ is metrizable due to Proposition 1.17. □

3.2 Banach-Mazur theorem

Theorem 3.6 (Banach-Mazur theorem). *The Banach space $C[0,1]$ is injectively universal for the class of all separable Banach spaces. That is, any separable Banach space X embeds isometrically into $C[0,1]$.*

Proof. The proof will be a direct consequence of the following two facts:

Fact 1. If X is a separable Banach space, then there exists a compact, Hausdorff and metrizable space K such that X embeds isometrically into C(K).

Indeed, we take $K := B_{X^*} = B_{X^*}[0,1]$ with the relative weak* topology (topology induced by the weak* topology of X^* on B_{X^*}). If X is a separable Banach space, then by Theorems 3.4 and 3.5 we know K is compact and metrizable. Moreover, by Lemma 3.1, K is also Hausdorff. If we now consider the mapping

$$\begin{aligned} j \colon X &\to C(K) \\ x &\mapsto j(x) \end{aligned}$$

where

$$\begin{aligned} j(x) \colon K &\to \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

(remember that this is the natural embedding of a Banach space in its second dual from Definition 1.41) it is easy to see that it is linear (note that $K \subset X^*$) and that

$$\|j(x)\| = \sup\{\|j(x)(f)\| : f \in K\} = \sup\{\|f(x)\| : f \in B_{X^*}\} = \|x\|$$

by using Hahn-Banach theorem (more precisely, Corollary 1.43). Thus, j is an isometric embedding from X into $C(K)$.

Fact 2. If K is a compact, Hausdorff and metrizable space, then $C(K)$ embeds isometrically into $C[0,1]$.

We split the proof of this statement into some steps:

1. If K is a compact, Hausdorff and metrizable space, then K embeds homeomorphically into $[0,1]^{\mathbb{N}}$ by Proposition 1.31.

2. If E is a closed subset of $[0,1]$, then $C(E)$ embeds isometrically into $C[0,1]$.
   To show this, we define a linear extension operator $A\colon C(E) \to C[0,1]$ so that $A(f)|_E = f$ for all $f \in C(E)$. Notice that as $[0,1]\backslash E$ is open in $[0,1]$, it can be written as a countable disjoint union of relatively open intervals. To see this, note that for any $x \in [0,1]\backslash E$ there must exist some open interval I such that $x \in I \subset [0,1]\backslash E$. If there exists one such interval, then there exists one "largest" interval which contains x (the union of all such intervals). Denote by $\{I_x\}_{x\in[0,1]\backslash E}$ the family of all such maximal intervals. These intervals I_x are pairwise disjoint (otherwise they would not be maximal) and every interval contains a rational number, so therefore there can only be a countable number of intervals in the family. Thus, we may extend f to be affine on each such interval interior to $[0,1]$ and to be constant on any such interval containing an endpoint of $[0,1]$. This procedure clearly gives an isometry as A is linear and preserves the norm (in E it behaves as the identity function an outside of E it is affine or constant).

3. We are ready now to complete the proof of Fact 2 and, therefore, of the theorem.
   Let ψ be the continuous surjection from Δ to $[0,1]^{\mathbb{N}}$ given on Proposition 1.27 and let us consider K as a closed subset of $[0,1]^{\mathbb{N}}$ (by Step 1 we are able to regard it as a subset of $[0,1]^{\mathbb{N}}$, which we now would be closed by Proposition 1.12). It follows that if $E := \psi^{-1}(K)$, then E is homeomorphic to a (closed) subset of $[0,1]$ (it is clear that E is a closed subset of Δ, and now by Proposition 1.25 this result follows). Then, by Step 2, $C(E)$ embeds isometrically into $C[0,1]$. Finally, $f \mapsto f \circ \psi$ embeds $C(K)$ isometrically into $C(E)$ and, therefore, $C(K)$ embeds isometrically into $C[0,1]$ after composition of both isometries.

□

Observation 3.7. Note that this proof also shows that X embeds isometrically into $C(\Delta)$. At first sight, both domains are equally useful, but in the extended version for Banach lattices just the Cantor Set will be valid.

Lastly, we present an example of two Banach lattices that aren't comparable, that is, none can be lattice-isometrically embedded into the other, to make evident the need of a stronger version of this theorem for Banach lattices in which another universal injetive space is used, and to justify the existence of our next section.

Example 3.8. The spaces $C[0,1]$ and $L_1[0,1]$ are not comparable being regarded as Banach lattices.

Proof. It suffices to show that the spaces $C(\mathcal{C}) \subset C[0,1]$ and $L_1[0,1]$ are not comparable being regarded as Banach lattices.

First, let us assume that there exists some Banach lattice isomorphism

$$T\colon C(\mathcal{C}) \to X \subset L_1[0,1]$$

and let us denote the (continuous) constant function in 1 defined in $\mathcal{C}$ as $\mathbb{1}$.

For a fixed $n > 0$, we divide the interval $[0,1]$ into equal subintervals of length $\frac{1}{3^n}$ where the k-th subinterval will be called I_k, for $k \in \{1, ..., 3^n\}$. The (continuous) constant function in 1 defined in $I_k \cap \mathcal{C}$ when the set is not empty will be denoted as $\mathbb{1}_k$. Note that these are 2^n positive and disjoint functions such that

$$\mathbb{1} = \sum_{k=1}^{2^n} \mathbb{1}_k.$$

Thus, as $\|x\|_{C(\mathcal{C})} \leq \left\|T^{-1}\right\|_{op} \cdot \|T(x)\|_{L_1}$ and $\left\|T^{-1}\right\|_{op} = \|T\|_{op}^{-1}$; we have

$$\|T(\mathbb{1})\|_{L_1} = \left\| T\left(\sum_{k=1}^{2^n} \mathbb{1}_k\right)\right\|_{L_1} = \left\|\sum_{k=1}^{2^n} T(\mathbb{1}_k)\right\|_{L_1} =$$
$$\sum_{k=1}^{2^n} \|T(\mathbb{1}_k)\|_{L_1} \geq \|T\|_{op} \cdot \sum_{k=1}^{2^n} \|\mathbb{1}_k\|_{C(\mathcal{C})} = 2^n \cdot \|T\|_{op}.$$

Since $\|T(\mathbb{1})\|_{L_1} \leq \|T\|_{op} \cdot \|\mathbb{1}\|_{C(\mathcal{C})} = \|T\|_{op} < \infty$, therefore making n tend to infinity we get $\|T(\mathbb{1})\|_{L_1} = \infty$, a contradiction.

Secondly, let us assume the existence of some lattice homomorphism

$$S\colon L_1[0,1] \to C(\mathcal{C})$$

and let us denote the constant function in 1 defined in $[0,1]$ as $\mathbb{1}$.

For a fixed $n > 0$, we divide the interval $[0,1]$ into equal subintervals of length $\frac{1}{n}$ where the k-th subinterval will be called I_k, for $k \in \{1, ..., n\}$. The constant function in 1 defined in I_k will be denoted as $\mathbb{1}_k$. Note that these are n positive and disjoint functions such that

$$\mathbb{1} = \bigvee_{k=1}^{n} \mathbb{1}_k.$$

Now since $\|S(x)\|_{C(\mathcal{C})} \leq \|S\|_{op} \cdot \|x\|_{L_1}$, it must be

$$\|S(\mathbb{1})\|_{C(\mathcal{C})} = \left\| S\left(\bigvee_{k=1}^{n} \mathbb{1}_k \right) \right\|_{C(\mathcal{C})} = \left\| \bigvee_{k=1}^{n} S(\mathbb{1}_k) \right\|_{C(\mathcal{C})} = \bigvee_{k=1}^{n} \|S\left(\mathbb{1}_k\right)\|_{C(\mathcal{C})} \leq \|S\|_{op} \cdot \frac{1}{n}.$$

Then making n tend to infinity we get $\|S(\mathbb{1})\|_{C(\mathcal{C})} = 0$, which is absurd because S being injective and linear would mean that $\mathbb{1} = 0$. $\square$

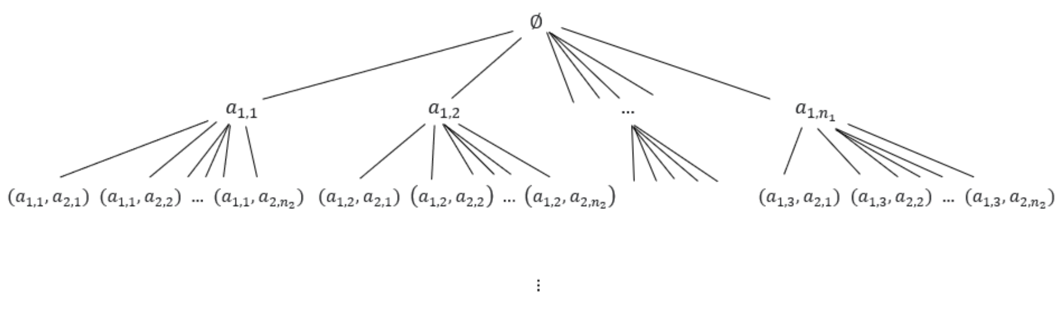
CHAPTER **4**

Universal injective Banach lattice for the class of all separable Banach lattices

In this chapter our main aim is to expose a proof for the first theorem that can be found in the article *Separable Universal Banach Lattices* by Leung-Li-Oikhberg-Tursi. For this, we first showed its analogous version for Banach spaces in Chapter 3.

4.1 Preliminaries

Definition 4.1. Let A_n, $n \in \mathbb{N}$, be finite non-empty sets and let $\hat{T}$ be $\bigcup_{k=0}^{\infty} \prod_{n=1}^{k} A_n$, where, as usual, the product $\prod_{n=1}^{k} A_n$ is defined to be $\emptyset$ if $k = 0$. Suppose that $\sigma = (a_1, ..., a_k) \in \prod_{n=1}^{k} A_n$; we say that σ has *length* k and write $|\sigma| = k$. For any $b \in A_{k+1}$, we denote the element $(a_1, ..., a_k, b) \in \prod_{n=1}^{k+1} A_n$ by (σ, b).



Example 4.2. The previous illustration shows a tree where $a_{i,j}$ is the j-th element in A_i, for $j = 1, ...n_i$.

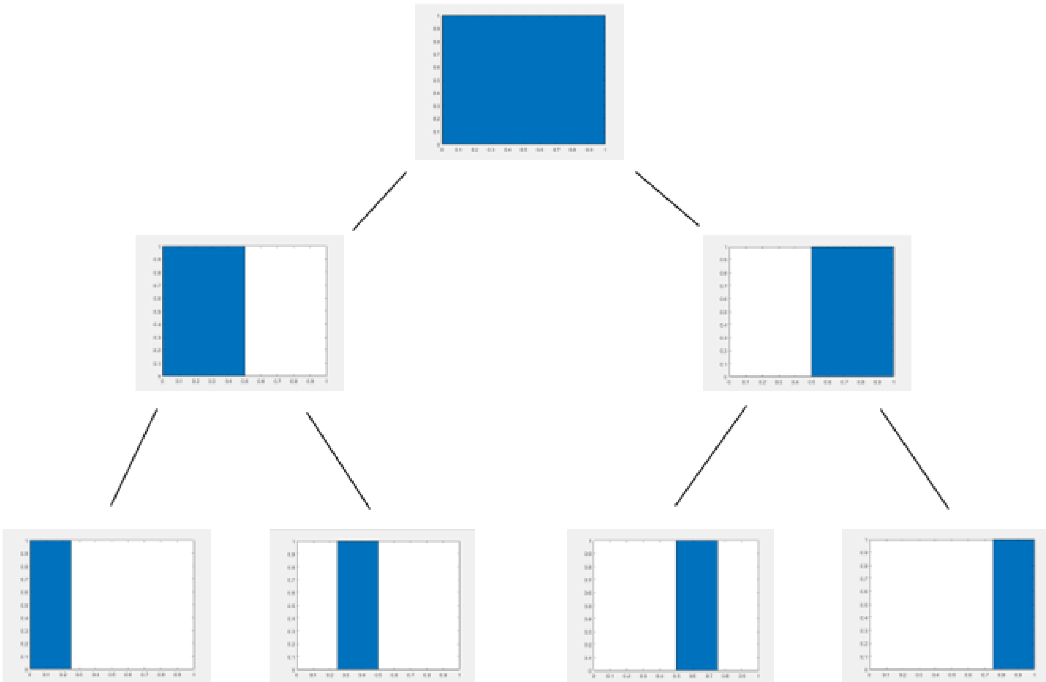
Definition 4.3. Let E be a Banach lattice. A family $(x_\sigma)_{\sigma \in \hat{T}}$ is said to be a *finitely branching tree* in E_+ if

(i) $x_\sigma \in E_+$ for all $\sigma \in \hat{T}$

(ii) For any $\sigma \in \hat{T}$ with $|\sigma| = k$, $(x_{(\sigma,b)})_{b \in A_{k+1}}$ is pairwise disjoint and

$$x_\sigma = \sum_{b \in A_{k+1}} x_{(\sigma,b)}.$$

Example 4.4. The following illustration shows the first three levels of a finitely branching tree where $E = C[0,1]$.



Observation 4.5. Note that the set $(x_{(\sigma,b)})_{b \in A_{k+1}}$ can be seen as a partition of x_σ by condition (ii). Thus, by induction, each level of a finitely branching tree (that is, $\{x_\sigma : |\sigma| = k\}$ for a certain $k \in \mathbb{N}$) must be a partition of $x_\emptyset$.

Lemma 4.6. *Let $(x_\sigma)_{\sigma \in \hat{T}}$ be a finitely branching tree in E_+, then $span\{x_\sigma : \sigma \in \hat{T}\}$ is a vector sublattice of E.*

Proof. We just need to see that for any $x, y \in S := span\{x_\sigma : \sigma \in \hat{T}\}$ it is $x \vee y \in S$ and $x \wedge y \in S$.

For any $x \in S$, then by Observation 4.5 one can derive that $x \in span\{x_\sigma : |\sigma| = n\}$ for a sufficiently large $n \in \mathbb{N}$. Analogously, for $y \in S$, we have $y \in span\{x_\sigma : |\sigma| = m\}$ for a certain $m \in \mathbb{N}$.

Without loss of generality, let $m \leq n$. Then both $x, y \in span\{x_\sigma : |\sigma| = n\}$ and we can express $x = \sum_{|\sigma|=n} c_\sigma \cdot x_\sigma$, $y = \sum_{|\sigma|=n} d_\sigma \cdot x_\sigma$ for some $c_\sigma, d_\sigma \in \mathbb{R}$.

Using Proposition 2.34, we have

$$x \vee y = \sum_{|\sigma|=n} (c_\sigma \vee d_\sigma) \cdot x_\sigma \text{ and } x \wedge y = \sum_{|\sigma|=n} (c_\sigma \wedge d_\sigma) \cdot x_\sigma,$$

and hence our proof is complete. □

4.2 Main result

Lemma 4.7. *Let $f, g \colon D \to \mathbb{R}$ be two continuous real functions with $0 \leq f(x) \leq g(x) \leq 1$, for every $x \in D$. The composition given by $\chi_{[f(x),g(x)]}$ for every $x \in D$ is a continuous function from D to L_1.*

Proof. The scheme

$$\begin{array}{ccccc} D & \xrightarrow{\quad i \quad} & \mathbb{R}^2 & \xrightarrow{\quad j \quad} & L_1 \\ \\ x & \dashrightarrow & (f(x), g(x)) & \dashrightarrow & \chi_{[f(x),g(x)]} \end{array}$$

is continuous because of being a composition of continuous functions. The function i is continuous because its components are continuous by hypothesis.

Now for any (x, y) and (x', y') in the image of i, we have

$$d_{L_1}\left(j(x,y), j(x',y')\right) = \left\|j(x,y) - j(x',y')\right\|_{L_1} = \left\|\chi_{[x,y]} - \chi_{[x',y']}\right\|_{L_1} =$$
$$\int_{\mathbb{R}} |\chi_{[x,y]}(t) - \chi_{[x',y']}(t)|dt \leq |x - x'| + |y - y'| \leq 2\sqrt{|x-x'|^2 + |y-y'|^2} =$$
$$2 \cdot \left\|(x,y) - (x',y')\right\|_2 = 2 \cdot d_2\left((x,y),(x',y')\right)$$

It is obvious that this gives that j is continuous an the result follows. □

Proposition 4.8. *Let E be a Banach lattice. Suppose that there is a finitely branching tree $(x_\sigma)_{\sigma \in \hat{T}}$ in E_+ so that E is the closed linear span of $(x_\sigma)_{\sigma \in \hat{T}}$. Then there exists a compact metric space K so that E is a lattice isometric to a closed sublattice of $C(K, L_1)$.*

Proof. Obviously, under the given assumption, E is a separable Banach lattice as the set $span\{x_\sigma : \sigma \in \hat{T}\}$ is dense in E by hypothesis and $\hat{T}$ is countable by definition.

Let K be the positive part of the closed unit ball of E^*, B_{E^*}, endowed with the relative weak* topology. Now we take $K^c = B_{E^*} \backslash K$ and see that it is open in B_{E^*}. Let $x^* \in K^c$, then it must be $x^*(x) < 0$ for some $x \geq 0$. For a small enough $\epsilon > 0$, if $|x^*(x) - y^*(x)| < \epsilon$ then also $y^*(x) < 0$ for that same $x \geq 0$. By taking $W_\epsilon(x^*; x) \cap B_{E^*}$ where $W_\epsilon(x^*; x)$ is a basic neighborhood of x^* on E^* endowed with the weak* topology described as on Definition 1.41, we obtain a relatively open neighborhood of x^* contained in K^c. Thus, K is a compact metrizable topological space by Proposition 1.6 and Theorems 3.4 and 3.5.

By rescaling if necessary, we may assume that $\|x_\emptyset\| \leq 1$.

For each $\sigma \in \hat{T}$, the function $g_\sigma \colon K \to \mathbb{R}$ given by $g_\sigma(x^*) = x^*(x_\sigma)$ is nonnegative (as every $x^* \in K$ and $x_\sigma \in E_+$) and continuous on K by definition of weak* topology. Furthermore, for all $\sigma \in \hat{T}$ with $|\sigma| = k$, it follows from property (ii) and the fact that every $x^* \in K$ is linear that

$$g_\sigma = \sum_{b \in A_{k+1}} g_{(\sigma,b)}. \tag{4.1}$$

We now define functions $f_\sigma \colon K \to L_1$ inductively, for every $\sigma \in \hat{T}$, as follows. Let $f_\emptyset(x^*) = \chi_{[0,g_\emptyset(x^*)]}$, and note that $\chi_{[0,g_\emptyset(x^*)]} \subset L_1[0,1]$, as $\|x_\emptyset\| \leq 1$ implies $g_\emptyset(x^*) = |x^*(x_\emptyset)| \leq 1$. By the continuity of $g_\emptyset$ and Lemma 4.7, we see that $f_\emptyset$ is a continuous function from K into L_1.

In general, assume that f_σ has been defined so that $f_\sigma(x^*) = \chi_{[c(x^*),d(x^*)]}$, where $c, d \colon K \to \mathbb{R}$ are nonnegative continuous functions so that $d - c = g_\sigma$. Label the elements in A_{k+1} as $b_1, ..., b_r$. Define $f_{(\sigma,b_i)}(x^*)$, $1 \leq i \leq r$, to be the characteristic function of the interval

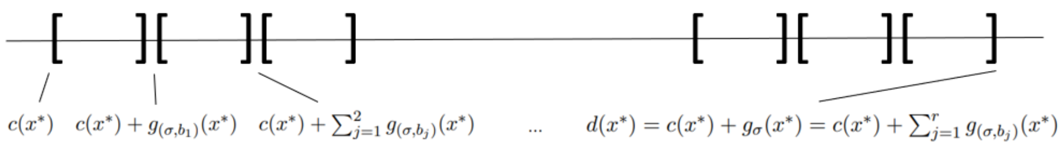
$$\left[c(x^*) + \sum_{j=1}^{i-1} g_{(\sigma,b_j)}(x^*), c(x^*) + \sum_{j=1}^{i} g_{(\sigma,b_j)}(x^*) \right].$$

By continuity of c and $g_{(\sigma,b_j)}$, and by Lemma 4.7, $f_{(\sigma,b_i)}$ is a continuous function from K into L_1 for each i. This completes the inductive definition of f_σ, for every $\sigma \in \hat{T}$.

Next step is to state the following:

$$f_\sigma = \sum_{b \in A_{k+1}} f_{(\sigma,b)} \text{ if } |\sigma| = k \tag{4.2}$$

(equality in the L_1 sense at each $x^* \in K$). To see this, note that for every $x^* \in K$ this is a sum of characteristic functions which are defined in disjoint intervals. Moreover, these intervals overlap leaving no space in between.



The previous illustration displays the idea, and follows from 4.1. Hence, the sum will result in the characteristic function defined in the total interval, that is, $\chi_{[c(x^*),d(x^*)]}$, which is $f_\sigma(x^*)$ as we wanted to show.

We see that the map $x_\sigma \mapsto f_\sigma$, for every $\sigma \in \hat{T}$, extends to a linear map u from $span\{x_\sigma : \sigma \in \hat{T}\}$ into $C(K, L_1)$. Let $x \in span\{x_\sigma : \sigma \in \hat{T}\}$, which by Observation 4.5 can be expressed as $x = \sum_{|\sigma|=k} c_\sigma x_\sigma$ for some $k \in \mathbb{N}$ and $c_\sigma \in \mathbb{R}$. We define $u(x)$ as $\sum_{|\sigma|=k} c_\sigma f_\sigma$. This is well defined because any other expression for x involving a different length k gives an equivalent expression for $u(x)$, by making use of (ii) and 4.2.

It remains to show that u is linear. To see this, we prove that the map preserves the sums. For any $x, y \in span\{x_\sigma : \sigma \in \hat{T}\}$, as before we are able to express $x = \sum_{|\sigma|=n} c_\sigma \cdot x_\sigma$ and $y = \sum_{|\sigma|=n} d_\sigma \cdot x_\sigma$ for sufficiently large $n \in \mathbb{N}$ and some $c_\sigma, d_\sigma \in \mathbb{R}$. Then clearly $u(x+y) = u(x) + u(y)$.

To see that the map u is a lattice homomorphism, we first check that if σ and τ are distinct elements in $\hat{T}$ of the same length, then $f_\sigma(x^*) \wedge f_\tau(x^*) = 0$ (in L_1). By induction on the length, for $|\sigma| = |\tau| = 0$ this is true by vacuity, as it would mean that $\sigma = \tau = \emptyset$. Now let us assume it holds for a length n and take $|\sigma| = |\tau| = n+1$.

- If $\sigma = (a_1, ..., a_n, b_i)$ and $\tau = (a_1, ..., a_n, b_j)$ for some $b_i, b_j \in A_{n+1}$, then f_σ and f_τ must be characteristic functions defined in disjoint intervals. Hence, clearly $f_\sigma(x^*) \wedge f_\tau(x^*) = 0$.

- If, on the other hand, $\sigma = (\hat{\sigma}, b_i)$ and $\tau = (\hat{\tau}, b_j)$ for some $b_i, b_j \in A_{n+1}$ where $\hat{\sigma}$ and $\hat{\tau}$ are distinct, then by the inductive hypothesis $f_{\hat{\sigma}}(x^*) \wedge f_{\hat{\tau}}(x^*) = 0$ and now by Observation 4.5 and (4.2) it is $f_\sigma(x^*) \wedge f_\tau(x^*) = 0$.

Now for any distinct $x, y \in span\{x_\sigma : \sigma \in \hat{T}\}$, we have $x = \sum_{|\sigma|=n} c_\sigma \cdot x_\sigma$ and $y = \sum_{|\sigma|=n} d_\sigma \cdot x_\sigma$ for sufficiently large $n \in \mathbb{N}$ and some $c_\sigma, d_\sigma \in \mathbb{R}$. Because of the previous observation we can use Proposition 2.34 on the set $\{f_\sigma : |\sigma| = n\}$ and state

$$u(x \wedge y) = u\left(\sum_{|\sigma|=n} (c_\sigma \wedge d_\sigma) \cdot x_\sigma\right) = \sum_{|\sigma|=n} (c_\sigma \wedge d_\sigma) \cdot f_\sigma = u(x) \wedge u(y).$$

Hence the fact that u preserves the infima follows. As u preserves the infima, it must preserve the suprema as $x \vee y = -(-x) \wedge (-y)$. Thus, because of the fact that $span\{x_\sigma : \sigma \in \hat{T}\}$ is a sublattice of E by Lemma 4.6, the map u is a lattice homomorphism.

Next, we show that u is an (into) isometry. Let $x \in span\{x_\sigma : \sigma \in \hat{T}\}$. Again by Observation 4.5 we write $x = \sum_{|\sigma|=k} c_\sigma x_\sigma$ for some $k \in \mathbb{N}$ and $c_\sigma \in \mathbb{R}$. Then $|x| = \sum_{|\sigma|=k} |c_\sigma| x_\sigma$ by Corollary 2.35 and

$$|u(x)| = u(|x|) = \sum_{|\sigma|=k} |c_\sigma| f_\sigma.$$

By construction, $\|f_\sigma(x^*)\|_{L_1} = g_\sigma(x^*) = x^*(x_\sigma)$. Since K is the positive part of the ball of E^* and applying Lemma 2.39 and Corollary 2.76, one can derive that

$$
\begin{aligned}
\|u(x)\| = \|\,|u(x)|\,\| &= \sup_{x^*\in K}\left\|\sum_{|\sigma|=k}|c_\sigma| f_\sigma(x^*)\right\|_{L_1} \\
&= \sup_{x^*\in K}\sum_{|\sigma|=k}|c_\sigma|\cdot\|f_\sigma(x^*)\|_{L_1} \\
&= \sup_{x^*\in K}\sum_{|\sigma|=k}|c_\sigma| x^*(x_\sigma) = \sup_{x^*\in K} x^*\left(\sum_{|\sigma|=k}|c_\sigma| x_\sigma\right) \\
&= \sup_{x^*\in K} x^*(|x|) = \|\,|x|\,\| \\
&= \|x\|\,.
\end{aligned}
$$

Hence u is a lattice isometry from $span\{x_\sigma : \sigma \in \hat{T}\}$ into $C(K, L_1)$.

As $span\{x_\sigma : \sigma \in \hat{T}\}$ is dense in E by assumption, we see that u extends to a lattice isometry from E into $C(K, L_1)$. As for any $x \in E$ there must exist a sequence $(x_n) \subset span\{x_\sigma : \sigma \in \hat{T}\}$ such that $\lim_n x_n = x$, we may define $u(x) = \lim_n u(x_n)$. For this, we need to show that such limit exists and does not depend on the chosen sequence. First, $u(x_n)$ is convergent because of being a Cauchy sequence inside a Banach lattice (u is an isometry on $span\{x_\sigma : \sigma \in \hat{T}\}$, so $\|u(x_n) - u(x_m)\| = \|u(x_n - x_m)\| = \|x_n - x_m\|$ where x_n is convergent by hypothesis and hence Cauchy). Also, given other sequence $(x'_n) \subset span\{x_\sigma : \sigma \in \hat{T}\}$ convergent to the same x, we would have $\|u(x_n) - u(x'_n)\| = \|u(x_n - x'_n)\| = \|x_n - x'_n\| \leq \|x_n - x\| + \|x'_n - x\|$ which must tend to zero. Thus, this is well defined and it is obvious that all the properties extend to u as the limit preserves them. □

Lemma 4.9. *Let (x_n) be a sequence in a Banach space. Then, if $\sum_{n=1}^{\infty}\|x_n\|$ converges, $\sum_{n=1}^{\infty} x_n$ also converges.*

Proof. By the triangle inequality we have, for any n and $p \geq 1$,

$$
\left\|\sum_{k=1}^{n+p} x_k - \sum_{k=1}^{n} x_k\right\| \leq \left|\sum_{k=1}^{n+p}\|x_k\| - \sum_{k=1}^{n}\|x_k\|\right|.
$$

By hypothesis the sequence $\sum_{k=1}^{n}\|x_k\|$ converges and is therefore a Cauchy sequence. By the inequality above, $\sum_{k=1}^{n} x_k$ is a Cauchy sequence and will converge to an element of the Banach lattice by completeness. □

Proposition 4.10. *Let E be a separable Banach lattice, regarded as a closed sublattice of its bidual E^{**}. There is a Banach lattice F such that $E \subseteq F \subseteq E^{**}$, F_+ contains a finitely branching tree $(x_\sigma)_{\sigma\in\hat{T}}$ and $span\{x_\sigma : \sigma \in \hat{T}\}$ is dense in F.*

Proof. Let $(e_i)_{i=1}^{\infty}$ be a countable dense subset of E consisting of nonzero vectors [1].

[1] Note that we can assume every vector to be nonzero because in case the null vector belongs to our set we can substitute it by the sequence $\{(\frac{1}{n})\}_n$ so that every vector that was close from 0 will also be close to a certain $\frac{1}{n}$ for a big enough $n \in \mathbb{N}$.

We shall construct recursively a finitely branching tree $(x_\sigma)_{\sigma\in\hat{T}}$ in E_+^{**} so that, for any $1\le m\le n$,

$$d(e_m, span\{x_\sigma : |\sigma| = n\}) < 2^{-n}.$$

Then the proposition follows by taking F to be the closed linear span of $(x_\sigma)_{\sigma\in\hat{T}}$ in E^{**}, as the previous condition guarantees that $E\subseteq F$. For any given $x\in E$ and $\epsilon>0$, there must exist some e_m such that $dist(x,e_m)<\epsilon$, but for a big enough n it is also

$$d(x, span\{x_\sigma : |\sigma| = n\}) \le d(x,e_m) + d(e_m, span\{x_\sigma : |\sigma| = n\}) < \epsilon$$

and hence x belongs to the closed linear span of $(x_\sigma)_{\sigma\in\hat{T}}$, that is, $x\in F$.

Start the construction by setting $A_0=\emptyset$ and

$$x_\emptyset = e = \sum_{i=1}^{\infty} \frac{|e_i|}{2^i\,\|e_i\|}.$$

This series is convergent as

$$\sum_{i=1}^{\infty}\left\|\frac{|e_i|}{2^i\,\|e_i\|}\right\| = \sum_{i=1}^{\infty}\frac{\|e_i\|}{|2^i|\,\|e_i\|} = \sum_{i=1}^{\infty}\frac{1}{2^i} = \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i = 1$$

(see Lemma 4.9).

Suppose that $n\in\mathbb{N}\cup\{0\}$ and the sets $A_0, A_1, ..., A_n$ and vectors $x_\sigma\in E_+^{**}$ $(|\sigma|\le n)$ have already been selected so that condition (ii) from the definition of finitely branching tree above is satisfied for every σ with $|\sigma|<n$. In particular, by Observation 4.5,

$$\sum_{|\sigma|=n} x_\sigma = e.$$

Since for all $1\le i\le n+1$,

$$|e_i| = 2^i\,\|e_i\|\cdot\frac{|e_i|}{2^i\,\|e_i\|} \le 2^i\,\|e_i\|\cdot\sum_{i=1}^{\infty}\frac{|e_i|}{2^i\,\|e_i\|} = 2^i\,\|e_i\|\cdot e,$$

e_i lies in the principal ideal generated by e in E^{**}.

By Freudenthal's Spectral Theorem 2.74 applied to $\epsilon = 2^{-(n+1)}$ and $x=e_m$, there exist mutually disjoint $z_1, ..., z_N\in E_+^{**}$ so that $z_1+...+z_N=e$, and

$$d(e_m, span\{z_1, ..., z_N\}) < 2^{-(n+1)}$$

for $1\le m\le n+1$.

Denote by P_i the band projection from E^{**} onto the band generated by z_i in E^{**}, $1\le i\le N$. Note that we can do so because of the fact that every bidual Banach lattice has the principal projection property (see 2.70).

Let $A_{n+1}=\{1,...,N\}$; for $\sigma\in\prod_{k=1}^n A_k$ and $i\in A_{n+1}$, let $x_{(\sigma,i)} = P_i(x_\sigma)$. Now by Proposition 2.65 we can argue that $\sum_{i=1}^N P_i$ is clearly the band projection onto B, where B

is the band generated by e in E^{**}, noting that $z_1 + ... + z_N = e$. Also, since x_σ lies in B (we just saw that $x_\sigma \leq e$), we have

$$x_\sigma = P_B(x_\sigma) = \sum_{i=1}^{N} P_i(x_\sigma) = \sum_{i \in An+1} x_{(\sigma,i)}.$$

This completes the inductive construction of $(x_\sigma)_{\sigma \in \hat{T}}$, where $\hat{T} = \bigcup_{k=0}^{\infty} \prod_{n=1}^{k} A_n$. Clearly, $(x_\sigma)_{\sigma \in \hat{T}}$ is a finitely branching tree in E_+.

Furthermore, in the notation above,

$$z_i = P_i(e) = \sum_{|\sigma|=n} P_i(x_\sigma) = \sum_{|\sigma|=n} x_{(\sigma,i)} ,$$

so $span\{z_1, ..., z_N\} \subseteq span\{x_\sigma : |\sigma| = n+1\}$. Thus, for $1 \leq m \leq n+1$,

$$d(e_m, span\{x_\sigma : |\sigma| = n+1\}) \leq d(e_m, span\{z_1, ..., z_N\}) < 2^{-(n+1)}.$$

□

Theorem 4.11. *The Banach lattice $C(\Delta, L_1)$ is injectively universal for the class of all separable Banach lattices. That is, any separable Banach lattice E embeds lattice isometrically into $C(\Delta, L_1)$.*

Proof. By Propositions 4.8 and 4.10, there are a compact metric space K and a lattice isometry u from E into $C(K, L_1)$ (note that F is lattice isometric to a closed sublattice of $C(K, L_1)$ and $E \subseteq F$). It is well known that there exists a continuous surjection $\pi \colon \Delta \to K$ (see Proposition 1.32). Then the map $j \colon E \to C(\Delta, L_1)$ given by $j(x) = u(x) \circ \pi$ is a lattice isometry as so it is u and π is a surjection:

$$\|j(x)\| = \|u(x) \circ \pi\| = \sup_{y \in \Delta} \|u(x)(\pi(y))\|_{L_1} = \sup_{z \in K} \|u(x)(z))\|_{L_1} = \|u(x)\| = \|x\| .$$

□

Observation 4.12. Recalling Example 3.8, it is not surprising that the resulting universal injective Banach lattice for the class of all separable Banach lattices is a combination of both $C[0,1]$ and $L_1[0,1]$.

Web Resources

Math.StackExchange

Mathematics Stack Exchange is a question and answer site for people studying math at any level and professionals in related fields.
`https://math.stackexchange.com/`

- Any open subset of $\mathbb{R}$ is a countable union of disjoint open intervals
- C be a closed subset of the Cantor set Δ. Show the existence of a continuous function $f \colon \Delta \to C$ s.t. $f(x) = x$, $x \in C$
- Cantor set: Lebesgue measure and uncountability
- Prove if X is a compact metric space, then X is separable
- Show that the countable product of metric spaces is metrizable
- The product of Hausdorff spaces is Hausdorff

ProofWiki

ProofWiki is an online compendium of mathematical proofs. Their goal is the collection, collaboration and classification of mathematical proofs.
`https://proofwiki.org/`

- Bijection is Open iff Closed
- Bijection is Open iff Inverse is Continuous
- Closed Subspace of Compact Space is Compact
- Compact Subspace of Hausdorff Space is Closed
- Continuous Bijection from Compact to Hausdorff is Homeomorphism

- Continuous Image of Compact Space is Compact
- Definition:Metrizable Topology
- T_2 Property is Hereditary

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